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EFDA-JET REPORT

Comparison of Coulomb collision rates in the plasma physics and magnetically confined fusion literature

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Abstract. Expressions for Coulomb collision rates in the magnetised plasma physics and nuclear fusion literature are reviewed. General collisional momentum and energy exchange rates are given for plasma particle species assumed to be in local thermodynamical equilibrium. Specific expressions for a simple plasma are then compared with those presented in numerous standard references. Differences by factors of order unity are found in several sources and are linked to the replacement of the reduced mass by the test particle mass at some stage of the derivation.

1. Introduction

Binary Coulomb collisions between charged particles, characterised by a cubic dependence of the collision rates on the relative particle velocity, are one of the distinguishing features of plasma physics and play a crucial role in a variety of transport, relaxation and dissipative phenomena in magnetically confined plasmas [1–20].

The problem of *test* particles of species s moving through a Maxwellian background of *field* particles of species s' was solved half a century ago [1–5]. The results are summarised in a variety of plasma physics sources, including review articles by Trubnikov [6] and Hinton [9], and are generally expressed in terms of four velocity-dependent relaxation rates associated with test particle *slowing down*,

$$\nu_{ss'}^{\mathrm{t}}(\upsilon) \equiv -\frac{1}{\mathrm{v}_{\parallel s}}\frac{\partial \mathrm{v}_{\parallel s}}{\partial t} = \frac{4\gamma_{ss'}n_{s'}}{m_s^2\mathrm{v}_{Ts'}^3}\left(1+\frac{m_s}{m_{s'}}\right)\frac{\Psi(\upsilon)}{\upsilon}, \tag{1}$$

deflection (or transverse diffusion),

$$\nu_{ss'}^{\mathrm{t},\perp}(\upsilon) \equiv \frac{2D_{ss'}^{\perp}}{\mathrm{v}_s^2} \equiv \frac{1}{\mathrm{v}_s^2}\frac{\partial \mathrm{v}_{\perp s}^2}{\partial t} = \frac{4\gamma_{ss'}n_{s'}}{m_s^2\mathrm{v}_{Ts'}^3}\frac{\Phi(\upsilon)-\Psi(\upsilon)}{\upsilon^3}, \tag{2}$$

dispersion (or parallel diffusion),

$$\nu_{ss'}^{\mathrm{t},\parallel}(\upsilon) \equiv \frac{D_{ss'}^{\parallel}}{\mathrm{v}_s^2} \equiv \frac{1}{\mathrm{v}_s^2}\frac{\partial \mathrm{v}_{\parallel s}^2}{\partial t} = \frac{4\gamma_{ss'}n_{s'}}{m_s^2\mathrm{v}_{Ts'}^3}\frac{\Psi(\upsilon)}{\upsilon^3}, \tag{3}$$

and *energy loss*,

$$\nu^{\mathrm{t},\varepsilon}_{ss'}(\upsilon) \equiv \frac{1}{K_s^2}\frac{\partial K_s^2}{\partial t} = \frac{16\gamma_{ss'}n_{s'}}{m_s^2 \mathrm{v}_{Ts'}^3}\frac{\Psi(\upsilon)}{\upsilon^3} \tag{4}$$

Note that due to conservation of energy, these four relaxation rates are related as follows,

$$\nu^{\mathrm{t},\varepsilon}_{ss'}(\upsilon) + \nu^{\mathrm{t},\perp}_{ss'}(\upsilon) + \nu^{\mathrm{t},\parallel}_{ss'}(\upsilon) = 2\nu^{\mathrm{t}}_{ss'}(\upsilon). \tag{5}$$

In the above, $K_s \equiv m_s \mathrm{v}_s^2/2$ is the test particle kinetic energy, $n_{s'}$, $\mathbf{V}_{s'}$ and $T_{s'}$ are the particle density, flow velocity and temperature of the field particles,

$$n_{s'} = \int f_{s'} d\mathbf{v}, \qquad n_{s'}\mathbf{V}_{s'} = \int \mathbf{v} f_{s'} d\mathbf{v}, \qquad \frac{3}{2}n_{s'}T_{s} = \int \frac{1}{2}m_{s'}|\mathbf{v}-\mathbf{V}_s|^2 f_{s'} d\mathbf{v}. \tag{6}$$

and υ is the test particle speed v_s normalised by the field particle thermal speed, $\mathrm{v}_{Ts'}$,

$$\upsilon \equiv \upsilon_{ss'} \equiv \frac{\mathrm{v}_s}{\mathrm{v}_{Ts'}}, \qquad \mathrm{v}_{Ts'}^2 \equiv \frac{2T_{s'}}{m_{s'}}. \tag{7}$$

The error function, $\Phi(\upsilon)$, and the Chandrasekhar function, $\Psi(\upsilon)$, are defined by

$$\Phi(\upsilon) = \frac{2}{\pi^{1/2}}\int_0^{\upsilon} e^{-\xi^2} d\xi, \qquad \Psi(\upsilon) = \frac{1}{2\upsilon^2}\left[\Phi - \upsilon\frac{d\Phi}{d\upsilon}\right] \tag{8}$$

and take on the following values in the limits of $\upsilon = 0$, 1 and ∞,

$$\lim_{\upsilon\to 0}\Phi(\upsilon) = \lim_{\upsilon\to 0}\frac{2\upsilon}{\pi^{1/2}} = 0, \qquad \Phi(1) \approx 0.8, \qquad \lim_{\upsilon\to\infty}\Phi(\upsilon) = 1, \tag{9}$$

$$\lim_{\upsilon\to 0}\Psi(\upsilon) = \lim_{\upsilon\to 0}\frac{2\upsilon}{3\pi^{1/2}} = 0, \qquad \Psi(1) \approx 0.2, \qquad \lim_{\upsilon\to\infty}\Psi(\upsilon) = \lim_{\upsilon\to\infty}\frac{1}{2\upsilon^2} = 0. \tag{10}$$

Here we have used the relation $d\Phi/d\upsilon = 2\pi^{-1/2}\exp(-\upsilon^2)$. The charge related constant,

$$\gamma_{ss'} \equiv \frac{e_s^2 e_{s'}^2 \ln\Lambda_{ss'}}{8\pi\varepsilon_0^2} = \gamma_{s's}, \tag{11}$$

contains the logarithm, $\ln\Lambda_{ss'}$, of the ratio of maximum, $r_{\max}$, and minimum, $r_{\min}$, scattering distances between test and field particles,

$$\ln\Lambda_{ss'} \equiv \ln\left(\frac{r_{\max}}{r_{\min}}\right), \tag{12}$$

which is known as the Coulomb logarithm. The minimum distance can be approximated as the larger of the deBroglie length and the classical distance of closest approach [REF],

$$r_{\min} \approx \max\left(\frac{\hbar}{2\mu_{ss'}\langle u\rangle}, \frac{e_s e_{s'}}{4\pi\varepsilon_0^2 \mu_{ss'}\langle u\rangle^2}\right), \tag{13}$$

where $\mathbf{u} = \mathbf{v}_s - \mathbf{v}_{s'}$ is the relative velocity, $\langle\cdot\rangle$ denotes an average over both f_s and $f_{s'}$ and $\mu_{ss'}$ is the reduced mass,

$$\mu_{ss'} \equiv \frac{m_s m_{s'}}{m_s + m_{s'}} = \mu_{s's}. \tag{14}$$

The maximum distance is typically approximated as the effective Debye length [REF],

$$r_{\max} \approx \lambda_D^{\mathrm{eff}} = \left(\frac{\varepsilon_0 T_s}{\sum_a n_a e_a^2}\right)^{1/2} \tag{15}$$

with the summation performed over all species a whose thermal velocity exceeds the average relative velocity, $\mathrm{v}_{Ta} > \langle u \rangle$. The *NRL Plasma Formulary* [15] suggests the following expressions for electron-electron collisions,

$$\ln\Lambda_{\mathrm{ee}} \approx 30 - \ln\left(n_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{-3/2}\right), \qquad T_{\mathrm{e}} < 10\,\mathrm{eV} \tag{16}$$

$$\approx 31 - \ln\left(n_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{-1}\right), \qquad T_{\mathrm{e}} > 10\,\mathrm{eV},$$

for electron-ion or ion-electron collisions,

$$\ln\Lambda_{\mathrm{ei}} = \ln\Lambda_{\mathrm{ie}} \approx 30 - \ln\left(n_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{-3/2} Z\right), \qquad T_{\mathrm{i}} m_{\mathrm{e}}/m_{\mathrm{i}} < T_{\mathrm{e}} < 10\,Z^2\,\mathrm{eV} \tag{17}$$

$$\approx 31 - \ln\left(n_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{-1}\right), \qquad T_{\mathrm{i}} m_{\mathrm{e}}/m_{\mathrm{i}} < 10\,Z^2\,\mathrm{eV} < T_{\mathrm{e}}$$

$$\approx 37 - \ln\left(n_{\mathrm{i}}^{1/2} T_{\mathrm{i}}^{-3/2} Z^2 A^{-1}\right), \qquad T_{\mathrm{e}} < Z T_{\mathrm{i}} m_{\mathrm{e}}/m_{\mathrm{i}},$$

and for ion-ion collisions between ion species i and i$'$ with charge states Z and Z' and atomic masses A and A',

$$\ln\Lambda_{\mathrm{ii'}} = \ln\Lambda_{\mathrm{i'i}} \approx 31 - \ln\left[\frac{ZZ'(A+A')}{AT_{\mathrm{i'}} + A'T_{\mathrm{i}}}\left(\frac{n_{\mathrm{i}}Z^2}{T_{\mathrm{i}}} + \frac{n_{\mathrm{i'}}Z'^2}{T_{\mathrm{i'}}}\right)^{1/2}\right] \tag{18}$$

The Coulomb logarithm has only a weak dependence on n_s, T_s and the combination of s and s', with typical values in the range of 15–20.

When the test particles are likewise Maxwellian and the thermal speed of the heavier species is comparable to or smaller than that of the lighter species, then for each combination of s and s', the four rates (1)–(4) reduce to a single characteristic collision frequency. This reduction has lead to some differences, typically by a factor of order unity, in the definition of this basic collisional rate in the topical literature. While such differences are generally benign provided accuracy to within a factor two is sufficient, they become unacceptable when higher levels of accuracy are required. Moreover, they can generate confusion and lead to errors when expressions obtained from different sources are combined, as is often the case. In this article, we examine the definitions of collision rates found in roughly twenty common references, and benchmark these against symmetrical expressions [8–10].

Below we present a brief derivation of Coulomb collision rates for Maxwellian test and field particles following the account given by Hinton [9] and Hazeltine and Waelbroeck [10]. The discussion is divided into exchange rates of momentum and energy, presented in sections 2 and 3, respectively, followed by a comparison with expressions found elsewhere in the literature in section 4. To prevent confusion, all expressions are given as collision rates, ν (units of inverse time) rather than collision times, τ (units of time); the two are always related as $\tau = 1/\nu$. Unless otherwise stated, SI units will be used throughout, except for the temperatures which are given in electron volts. To convert any SI (mks) expression to Gaussian (cgs) units, simply replace ε_0 by $1/4\pi$, μ_0 by $4\pi/c^2$ and $\mathbf{B}$ by $\mathbf{B}/c$.

2. Collisional momentum exchange

The *collision rate* between *test* particles, s, and *field* particles, s', is typically defined as the time required to change the direction of the test particles by a right angle from their initial flow direction. Hence, the term collision rate tacitly assumes *momentum exchange*, unless the exchange of some other quantity, e.g. energy, is explicitly stated.

2.1. General momentum collision rates

In analogy with the slowing down rate of a single test particle species, given in (1), we define $\nu_{ss'}$ as the rate of change of the fluid momentum, which can be expressed as the friction force, $\mathbf{F}_{ss'}$, resulting from successive binary collisions between species s and s',

$$\mathbf{F}_{ss'} \equiv \int m_s \mathbf{v} C_{ss'}(f_s, f_{s'}) d\mathbf{v} = -m_s n_s \nu_{ss'} (\mathbf{V}_s - \mathbf{V}_{s'}) . \tag{19}$$

This friction force appears in the momentum equation for the plasma species s,

$$m_s n_s \left(\frac{\partial}{\partial t} + \mathbf{V}_s \cdot \nabla \right) \mathbf{V}_s = -\nabla p_s - \nabla \cdot \Pi_s + e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) + \sum_{s'} \mathbf{F}_{ss'}, \tag{20}$$

where n_s, $\mathbf{V}_s$ and T_s are the particle density, flow velocity and temperature of species s,

$$n_s = \int f_s d\mathbf{v}, \qquad n_s \mathbf{V}_s = \int \mathbf{v} f_s d\mathbf{v}, \qquad \frac{3}{2} n_s T_s = \int \frac{1}{2} m_s |\mathbf{v} - \mathbf{V}_s|^2 f_s d\mathbf{v}, \tag{21}$$

and Π_s is the deviation of the stress tensor from the scalar pressure $p_s = n_s T_s$. Hence, $\nu_{ss'}$ measures the deceleration of the flow of species s due to collisions with species s',

$$\frac{d\mathbf{V}_s}{dt} = -\sum_{s'} \nu_{ss'} (\mathbf{V}_s - \mathbf{V}_{s'}) , \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}_s \cdot \nabla, \tag{22}$$

where we neglected all other terms in (20). Note that $\mathbf{F}_{ss'}$ involves a velocity space integral of the Coulomb collision operator, $C_{ss'}(f_s, f_{s'})$, such that $\nu_{ss'}$ depends on the velocity distributions of both the test, $f_s(\mathbf{v})$, and field, $f_{s'}(\mathbf{v})$, particles.

To obtain an explicit expression for $\nu_{ss'}$, one can average (1) over the shifted Maxwellian test particle distribution, $f_s = f_{sM}(\mathbf{v} - \mathbf{V}_s)$; as will be demonstrated shortly, it is not sufficient to simply insert $\mathrm{v}_s = \mathrm{v}_{Ts}$ into (1), as the non-linearities in the integral generate errors of order unity. Alternatively, the desired result may be obtained by direct calculation of (19) from some approximation of the Coulomb collision operator $C_{ss'}$, often taken to be the large $\ln \Lambda_{ss'}$ limit known as the *Landau-Boltzmann* operator,

$$C_{ss'} = \frac{\gamma_{ss'}}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \int f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \left(\frac{\mathsf{I}}{u} - \frac{\mathbf{uu}}{u^3} \right) \cdot \chi_{ss'}(\mathbf{v}, \mathbf{v}') d\mathbf{v}' \tag{23}$$

where $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ is the relative velocity, $u = |\mathbf{u}|$ is the relative speed, I is the unit dyadic and

$$\chi_{ss'}(\mathbf{v}, \mathbf{v}') = \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \ln f_s - \frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} \ln f_{s'} \tag{24}$$

Due to the grazing nature of Coulomb collisions in the large $\ln \Lambda_{ss'}$ limit, the above collision operator can be recast into the well known *Fokker-Planck* form,

$$C_{ss'} = -\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A}_{ss'} f_s - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathsf{D}_{ss'} f_s) \right] = -\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{A}_{ss'} f_s) + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \mathsf{D}_{ss'} f_s \tag{25}$$

which describes the cumulative effect of multiple small angle deflections. The vector $\mathbf{A}_{ss'}$ and the tensor $\mathsf{D}_{ss'}$, often denoted as $\langle \Delta \mathbf{v} \rangle_{ss'} / \Delta t$ and $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle_{ss'} / \Delta t$, represent *dynamical friction* and *velocity space diffusion*, respectively, and are clearly related to the slowing down, deflection and dispersion rates given by (1), (2) and (3).

In order to calculate $C_{ss'M}$, it is useful to introduce the so-called *Rosenbluth* potentials,

$$g_{s'}(\mathbf{v}) = \int u f_{s'}(\mathbf{v}')d\mathbf{v}', \qquad h_{s'}(\mathbf{v}) = \int \frac{f_{s'}(\mathbf{v}')}{u}d\mathbf{v}', \tag{26}$$

and express $\mathbf{A}_{ss'}$ and $\mathsf{D}_{ss'}$ as velocity space gradients of these potentials,

$$\mathbf{A}_{ss'} = 2\frac{\gamma_{ss'}}{m_s^2}\left(1+\frac{m_s}{m_{s'}}\right)\frac{\partial h_{s'}}{\partial \mathbf{v}}, \qquad \mathsf{D}_{ss'} = 2\frac{\gamma_{ss'}}{m_s^2}\frac{\partial^2 g_{s'}}{\partial \mathbf{v}\partial \mathbf{v}}. \tag{27}$$

The Rosenbluth potentials are easily evaluated for Maxwellian field particles with zero flow velocity, $\mathbf{V}_{s'} = 0$, and finite temperature, $T_{s'}$,

$$g_{s'M}(\mathbf{v}) = \frac{n_{s'}\mathrm{v}_{Ts'}}{2\upsilon}\left[\upsilon\frac{d\Phi}{d\upsilon}+(1+2\upsilon^2)\Phi(\upsilon)\right], \qquad h_{s'M}(\mathbf{v}) = \frac{n_{s'}}{\mathrm{v}_{Ts'}}\frac{\Phi(\upsilon)}{\upsilon}. \tag{28}$$

The dynamical friction vector and velocity space diffusion tensor can now be evaluated as

$$\mathbf{A}_{ss'M} = \frac{2\gamma_{ss'}n_{s'}}{m_{s'}}\frac{\mathbf{v}}{\mathrm{v}^3}\left[\Phi(\upsilon)-\upsilon\frac{d\Phi}{d\upsilon}\right], \tag{29}$$

$$\mathsf{D}_{ss'M} = \frac{\gamma_{ss'}n_{s'}T_{s'}}{m_s^2 m_{s'}}\frac{1}{\mathrm{v}^3}\left[\mathsf{I}F_1(\upsilon)+3\frac{\mathbf{vv}}{\mathrm{v}^2}F_2(\upsilon)\right], \tag{30}$$

where

$$F_1(\upsilon) = \upsilon\frac{d\Phi}{d\upsilon}+(2\upsilon^2-1)\Phi(\upsilon), \tag{31}$$

$$F_2(\upsilon) = \left(1-\frac{2}{3}\upsilon^2\right)\Phi(\upsilon)-\upsilon\frac{d\Phi}{d\upsilon}. \tag{32}$$

Inserting (28) into (27), differentiating with respect to $\mathbf{v}$, and substituting the results into (25) yields the collision operator for Maxwellian test and field particles with zero relative flow velocity, $\mathbf{V}_s = \mathbf{V}_{s'}$, but potentially different particle densities and temperatures, [4–6, 9, 10],

$$C_{ss'M} = -\frac{2\gamma_{ss'}n_{s'}}{m_s m_{s'}\mathrm{v}_{Ts'}\mathrm{v}_{Ts}^2}\left(1-\frac{T_{s'}}{T_s}\right)\left[\frac{\Phi}{\upsilon}-\left(1+\frac{m_{s'}T_s}{m_s T_{s'}}\right)\frac{d\Phi}{d\upsilon}\right]f_{sM}. \tag{33}$$

As expected $C_{ss'M}$ vanishes when $T_s = T_{s'}$.

To evaluate the friction force, $\mathbf{F}_{ss'}$, the collision integral in (19) is performed with the above expression for $C_{ss'M}$, but with the assumption of a small but finite relative flow velocity between the field and test particles, $|\mathbf{V}_s - \mathbf{V}_{s'}| \ll \mathrm{v}_{Ts}$. Typically, the calculation is performed in the frame of reference of the field particles, such that $f_{s'} = f_{s'M}(\mathbf{v})$ and $f_s = f_{sM}(\mathbf{v}-\mathbf{V}_s)$. Expanding the latter in the small ratio V_s/v_{Ts} and computing the integrals yields the desired collision rate [9, 10],

$$\nu_{ss'} = \frac{8\gamma_{ss'}n_{s'}}{3\pi^{1/2}\mu_{ss'}m_s(\mathrm{v}_{Ts}^2+\mathrm{v}_{Ts'}^2)^{3/2}}. \tag{34}$$

It should be noted that Spitzer derived a general expression for the momentum exchange rate but in his result only the thermal velocity of the field particles appeared in the collision rate [REF]. Substituting $\gamma_{ss'}$ from (11) into (34) yields

$$\nu_{ss'} = \frac{2^{1/2}n_{s'}Z_s^2 Z_{s'}^2 e^4 \ln\Lambda_{ss'}}{12\pi^{3/2}\varepsilon_0^2\mu_{ss'}m_s[(T_s/m_s)+(T_{s'}/m_{s'})]^{3/2}}, \tag{35}$$

Table 1. The ratio of the velocity-dependent slowing down, (1), and energy loss, (4), rates evaluated with $\mathrm{v}_s = \mathrm{v}_{Ts}$, with the momentum, (34), and energy, (53), exchange rates for Maxwellian test and field particles. Note that the errors are twice in the latter case since the integrals involve higher powers of velocity.

	ei	ie	ee, ii
$\nu^{\mathrm{t}}_{ss'}(\upsilon_T)/\nu_{ss'}$	$3\pi^{1/2}/4 \approx 1.33$	1	$3(2\pi)^{1/2}\Psi(1) \approx 1.5$
$\nu^{\mathrm{t},\varepsilon}_{ss'}(\upsilon_T)/\nu^{\varepsilon}_{ss'}$	$3\pi^{1/2}/2 \approx 2.66$	2	$6(2\pi)^{1/2}\Psi(1) \approx 3$

where $Z_s = e_s/e$ and $Z_{s'} = e_{s'}/e$ are the charge states of species s and s', respectively. Alternatively, writing out the reduced mass in (14) gives the following explicit form

$$\nu_{ss'} = \frac{2^{1/2} n_{s'} Z_s^2 Z_{s'}^2 e^4 \ln\Lambda_{ss'}}{12\pi^{3/2}\varepsilon_0^2 m_s^{1/2} T_s^{3/2}} \frac{1 + m_s/m_{s'}}{[1 + (T_{s'}/m_{s'})/(T_s/m_s)]^{3/2}}. \tag{36}$$

Note from (35) that $n_s m_s \nu_{ss'} = n_{s'} m_{s'} \nu_{s's}$, consistent with momentum conservation in Coulomb collisions, which requires that $\mathbf{F}_{ss'} + \mathbf{F}_{s's} = \mathbf{0}$. However, the ratio $\nu_{ss'}/\nu_{s's} = (n_s/n_{s'})(m_{s'}/m_s)$ shows that for comparable particle densities but disparate masses, the fluid of light particles is subject to much stronger scattering than the fluid of heavy particles.

It is interesting to compare the above result with the approximate expression obtained by evaluating the velocity-dependent slowing down rate, (1), at the test particles thermal speed, i.e. at $\mathrm{v}_s = \mathrm{v}_{Ts}$,

$$\frac{\nu^{\mathrm{t}}_{ss'}(\upsilon_T)}{\nu_{ss'}} = \frac{3}{2}\pi^{1/2}(1+\upsilon_T^2)\frac{\Psi(\upsilon_T)}{\upsilon_T}, \qquad \upsilon_T \equiv \frac{\mathrm{v}_{Ts}}{\mathrm{v}_{Ts'}}. \tag{37}$$

This ratio is equal to unity for ion-electron collisions, to $3\pi^{1/2}/4 \approx 1.33$ for electron-ion collisions and to $3(2\pi)^{1/2}\Psi(1) \approx 1.5$ for like-particle collisions, see table 1.

2.2. Momentum collision rates for a simple plasma

Let us write down the various combinations of test, s, and field, s', particle species explicitly for a so-called *simple plasma*, that is, for a plasma with a single ionic particle species. Quasi-neutrality, expressed by $n_\mathrm{e} = Zn_\mathrm{i}$, will be assumed in the following; in a multiple ion species plasma, it can be shown that Z in subsequent expressions should be replaced by the *effective charge state*, Z_eff,

$$n_\mathrm{e} Z_\mathrm{eff} \equiv \sum_\mathrm{i} n_\mathrm{i} Z_\mathrm{i}^2. \tag{38}$$

Selecting $s = \mathrm{e}$ and $s' = \mathrm{i}$, we obtain the *electron-ion collision rate*, ν_ei, by inserting $Z_\mathrm{e} = 1$, $Z_\mathrm{i} = Z$, $m_\mathrm{i} \gg m_\mathrm{e}$ and thus $\mu_\mathrm{ei} = m_\mathrm{e}$ into (35) or (36),

$$\nu_\mathrm{ei} = \frac{2^{1/2} n_\mathrm{i} Z^2 e^4 \ln\Lambda_\mathrm{ei}}{12\pi^{3/2}\varepsilon_0^2 m_\mathrm{e}^{1/2} T_\mathrm{e}^{3/2}} = \frac{2^{1/2} n_\mathrm{e} Z e^4 \ln\Lambda_\mathrm{ei}}{12\pi^{3/2}\varepsilon_0^2 m_\mathrm{e}^{1/2} T_\mathrm{e}^{3/2}}. \tag{39}$$

In the above expression we have assumed $T_\mathrm{i}/T_\mathrm{e} \ll m_\mathrm{i}/m_\mathrm{e}$, and thus excluded the case when the ions are much hotter than the electrons such that their thermal speeds become comparable.

The complementary *ion-electron collision rate*, ν_{ie}, is found by setting $s = \mathrm{i}$, $s' = \mathrm{e}$, and thus inserting $Z_{\mathrm{e}} = 1$, $Z_{\mathrm{i}} = Z$, $m_{\mathrm{i}} \gg m_{\mathrm{e}}$ and $\mu_{\mathrm{ei}} = m_{\mathrm{e}}$ into (35),

$$\nu_{\mathrm{ie}} = \frac{2^{1/2} m_{\mathrm{e}}^{1/2} n_{\mathrm{e}} Z^2 e^4 \ln\Lambda_{\mathrm{ie}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{i}} T_{\mathrm{e}}^{3/2}} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right) Z \nu_{\mathrm{ei}}. \tag{40}$$

Note that the two rates differ by the product of the ratio of electron and ion masses, which is much smaller than one, and by the ratio of electron and ion densities, which is equal to Z. Moreover, by virtue of quasi-neutrality, $m_{\mathrm{e}} n_{\mathrm{e}} \nu_{\mathrm{ei}} = m_{\mathrm{i}} n_{\mathrm{i}} \nu_{\mathrm{ie}}$, momentum conservation is fulfilled under these approximations.

Let us next consider like-particle collisions. Selecting $s = s' = \mathrm{e}$, for which $\mu_{\mathrm{ee}} = m_{\mathrm{e}}/2$ and $Z_{\mathrm{e}} = 1$, we obtain the *electron-electron collision rate*, ν_{ee},

$$\nu_{\mathrm{ee}} = \frac{n_{\mathrm{e}} e^4 \ln\Lambda_{\mathrm{ee}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{3/2}} = \frac{\nu_{\mathrm{ei}}}{2^{1/2} Z}\frac{\ln\Lambda_{\mathrm{ee}}}{\ln\Lambda_{\mathrm{ei}}} \approx \frac{\nu_{\mathrm{ei}}}{2^{1/2} Z}, \tag{41}$$

which differs from ν_{ei} only by the ratio of ion to electron densities, which is equal to $1/Z$, and the factor of $1/2^{1/2}$, which comes from a combination of the reduced mass and the sum of thermal velocities in (34). Similarly, the *ion-ion collision rate*, ν_{ii}, in a plasma composed of a single ion species, is found by setting $s = s' = \mathrm{i}$, and thus inserting $\mu_{\mathrm{ii}} = m_{\mathrm{i}}/2$ and $Z_{\mathrm{i}} = Z$ into (35),

$$\nu_{\mathrm{ii}} = \frac{n_{\mathrm{i}} Z^4 e^4 \ln\Lambda_{\mathrm{ii}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{i}}^{1/2} T_{\mathrm{i}}^{3/2}} = \frac{n_{\mathrm{e}} Z^3 e^4 \ln\Lambda_{\mathrm{ii}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{i}}^{1/2} T_{\mathrm{i}}^{3/2}} \approx \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \left(\frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right)^{3/2} \frac{Z^2}{2^{1/2}} \nu_{\mathrm{ei}}. \tag{42}$$

We find that ν_{ii} differs from ν_{ei} by the product of the square root of the mass ratio, the 3/2 power of the temperature ratio, the square of the ion charge state and the factor of $1/2^{1/2}$, which once again originates from a combination of reduced mass and the sum of thermal velocities in (34). The relations between the four collision rates can be conveniently summarised as follows,

$$\nu_{\mathrm{ie}} : \nu_{\mathrm{ii}} : \nu_{\mathrm{ee}} : \nu_{\mathrm{ei}} \approx \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right) Z : \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \left(\frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right)^{3/2} \frac{Z^2}{2^{1/2}} : \frac{1}{2^{1/2} Z} : 1, \tag{43}$$

clearly expressing the efficient scattering of the light electrons by the heavy ions. The approximate sign reflects small differences in the Coulomb logarithms, (16)–(18).

2.3. Collision rates and transport coefficients

Following Braginskii [11], we define the *electron collision rate*, ν_{e}, simply as

$$\nu_{\mathrm{e}} \equiv \nu_{\mathrm{ei}} = \frac{2^{1/2} n_{\mathrm{i}} Z^2 e^4 \ln\Lambda_{\mathrm{ei}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{e}}^{1/2} T_{\mathrm{e}}^{3/2}}, \tag{44}$$

and the *ion collision rate*, ν_{i}, as

$$\nu_{\mathrm{i}} \equiv \nu_{\mathrm{ii}} = \frac{n_{\mathrm{i}} Z^4 e^4 \ln\Lambda_{\mathrm{ii}}}{12\pi^{3/2}\varepsilon_0^2 m_{\mathrm{i}}^{1/2} T_{\mathrm{i}}^{3/2}}. \tag{45}$$

These rates are of particular importance in the calculation of classical transport coefficients for a magnetised plasma; note that only transport parallel ($\|$) and perpendicular ($\perp$) to the magnetic field is affected, and that diamagnetic ($\wedge$) transport is independent of Coulomb collisions. For instance, the heat diffusivities, $\chi_{\parallel s}$, $\chi_{\perp s}$ and $\chi_{\wedge s}$, can be expressed as

$$\chi_{\parallel s} = \frac{C_{\parallel s} T_s}{\nu_s m_s}, \qquad \chi_{\perp s} = \frac{C_{\perp s} T_s \nu_s}{\Omega_s^2 m_s}, \qquad \chi_{\wedge s} = \frac{C_{\wedge s} T_s}{\Omega_s m_s}, \tag{46}$$

where $\Omega_s = Z_s eB/m_s$ is the gyro-frequency and ν_s is given by (44) or (45). The pre-factor $C_{\parallel \mathrm{e}}$, first calculated by Spitzer and Härm [7], is a weak function of ion charge with $C_{\mathrm{e}} = (3.197, 4.916, 6.972, 10.63)$ for $Z = (1, 2, 4, 16)$; to within 2% accuracy it can be approximated as

$$C_{\parallel \mathrm{e}} = 3.153 + 2.7005 \ln Z. \tag{47}$$

The other pre-factors were calculated by Braginskii [11], who expressed the diffusivities in terms of $\tau_{\mathrm{e}} = \nu_{\mathrm{e}}^{-1}$ and $\tau_{\mathrm{i}} = \nu_{\mathrm{i}}^{-1}$,

$$C_{\parallel \mathrm{i}} = 3.91, \qquad C_{\perp \mathrm{e}} = 4.7, \qquad C_{\perp \mathrm{i}} = 2.0, \qquad C_{\wedge s} = \frac{5}{2}. \tag{48}$$

Evidently, inaccuracies in the definition of ν_s would result in errors in the classical transport coefficients, such as (46), and thus the associated neoclassical transport coefficients [17].

3. Collisional energy exchange

The collisional exchange of thermal energy due to Coulomb collisions can be calculated in a manner similar to the momentum exchange. The general expression for Maxwellian plasma species was first given by Spitzer [1].

3.1. General energy collision rates

The energy conservation equation for a particle species s can be written as

$$\frac{3}{2}\left(\frac{\partial}{\partial t} + \mathbf{V}_s \cdot \nabla\right) p_s + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s + \Pi_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = \sum_{s'} W_{ss'}, \tag{49}$$

where $\mathbf{q}_s$ is the heat flux vector and $W_{ss'}$ is the *collisional thermal energy exchange rate* between particle species s and s',

$$W_{ss'} \equiv \int \frac{1}{2} m_s (\mathbf{v} - \mathbf{V}_s)^2 C_{ss'}(f_s, f_{s'}) d\mathbf{v}. \tag{50}$$

From the collision operator in (33), valid in the large $\ln \Lambda_{ss'}$ limit, we find that the collision rate for Maxwellian test and field particles is given by [1, 9, 10]

$$W_{ss'} = \frac{8}{\pi^{1/2}} \frac{\gamma_{ss'} n_s n_{s'} (T_{s'} - T_s)}{m_s m_{s'} (\mathrm{v}_{Ts}^2 + \mathrm{v}_{Ts'}^2)^{3/2}} = 3 \left(\frac{m_s}{m_s + m_{s'}}\right) \nu_{ss'} n_s (T_{s'} - T_s). \tag{51}$$

We thus define the *thermal equilibration rate*, or simply the *thermalisation rate*, $\nu_{ss'}^{\varepsilon}$, as the rate at which T_s and $T_{s'}$ converge to a common value as a result of Coulomb collisions,

$$\frac{dT_s}{dt} \equiv \sum_{s'} \nu_{ss'}^{\varepsilon} (T_{s'} - T_s), \tag{52}$$

where we have neglected all other terms in (49). Inserting (51) into (49) and retaining only the temporal derivative on the left-hand side yields

$$\nu^{\varepsilon}_{ss'} = \frac{16}{3\pi^{1/2}} \frac{\gamma_{ss'} n_{s'}}{m_s m_{s'} (\mathrm{v}^2_{Ts} + \mathrm{v}^2_{Ts'})^{3/2}} = 2\left(\frac{m_s}{m_s + m_{s'}}\right)\nu_{ss'}. \tag{53}$$

It is thus clear that for a mass ratio of order unity the momentum and energy exchange rates are comparable, while for disparate masses the energy change of the fluid of light particles is much smaller than its momentum exchange. Also note that $W_{ss'} + W_{s's} = 0$, consistent with energy conservation in Coulomb collisions.

For like-particle collisions, $s = s'$, the energy collision rate ν^{ε}_{ss} does not represent the rate of heat exchange between two Maxwellian fluids since, by (51), $W_{ss} = 0$. Rather, it measures the rate at which species s approach local thermodynamical equilibrium, i.e. the rate at which their velocity distribution relaxes to a Maxwellian. In this sense, the term *thermalisation rate* is better suited to ν^{ε}_{ss}, and *thermal equilibration rate* to $\nu^{\varepsilon}_{ss'}$ with $s \neq s'$. It is worth noting that the former is identical to the momentum exchange rate, ν_{ss}.

As before, we compare the above result with the approximate expression obtained by evaluating the velocity-dependent energy loss rate, (4), at the test particle thermal speed,

$$\frac{\nu^{\mathrm{t},\varepsilon}_{ss'}(\upsilon_T)}{\nu^{\varepsilon}_{ss'}} = 2\left(\frac{T_{s'}}{T_s}\right)\frac{\nu^{\mathrm{t}}_{ss'}(\upsilon_T)}{\nu_{ss'}} = 3\pi^{1/2}\left(\frac{T_{s'}}{T_s}\right)(1+\upsilon_T^2)\frac{\Psi(\upsilon_T)}{\upsilon_T} \tag{54}$$

The discrepancy between the two rates, represented by the above ratio, is roughly twice as large for energy exchange as for momentum exchange. For $T_s = T_{s'}$ it is equal to 2 for ion-electron collisions, $3\pi^{1/2}/2 \approx 2.66$ for electron-ion collisions and approximately $0.6(2\pi)^{1/2} \approx 3$ for like-particle collisions. This is consistent with the fact that $\nu^{\varepsilon}_{ss'}/\nu_{ss'}$ given by (53) differs by a factor of $2T_{s'}/T_s$ from the ratio of energy loss (4) and slowing down (1) rates of a test particle traveling at $\mathrm{v}_s = \mathrm{v}_{Ts}$,

$$\frac{\nu^{\mathrm{t},\varepsilon}_{ss'}(\upsilon_T)}{\nu^{\mathrm{t}}_{ss'}(\upsilon_T)} = 4\left(\frac{m_{s'}}{m_s + m_{s'}}\right)\frac{1}{\upsilon_T^2} = 4\left(\frac{T_{s'}}{T_s}\right)\left(\frac{m_s}{m_s + m_{s'}}\right) \tag{55}$$

As for the momentum relaxation rates, we find that estimates of thermal energy relaxation rates from the velocity dependent expressions at the test particle thermal velocity speed leads to errors of order unity.

3.2. Energy collision rates for a simple plasma

Once again, let us consider the various combinations of s and s' for a quasi-neutral simple plasma. The *electron-ion*, $\nu^{\varepsilon}_{\mathrm{ei}}$, and *ion-electron*, $\nu^{\varepsilon}_{\mathrm{ie}}$, *equilibration rates* are thus found as

$$\nu^{\varepsilon}_{\mathrm{ei}} = 2\left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)\nu_{\mathrm{ei}} = \frac{2^{1/2} m_{\mathrm{e}}^{1/2} n_{\mathrm{e}} Z e^4 \ln\Lambda_{\mathrm{ei}}}{6\pi^{3/2}\varepsilon_0^2 m_{\mathrm{i}} T_{\mathrm{e}}^{3/2}}, \tag{56}$$

$$\nu^{\varepsilon}_{\mathrm{ie}} = 2\nu_{\mathrm{ie}} = 2\left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right) Z\nu_{\mathrm{ei}} = Z\nu^{\varepsilon}_{\mathrm{ei}}. \tag{57}$$

Similarly, the *electron-electron*, $\nu^{\varepsilon}_{\mathrm{ee}}$, and *ion-ion*, $\nu^{\varepsilon}_{\mathrm{ii}}$, *thermalisation rates* are found as

$$\nu^{\varepsilon}_{\mathrm{ee}} = \nu_{\mathrm{ee}} = \nu_{\mathrm{e}}, \tag{58}$$

$$\nu_{\rm ii}^{\varepsilon} = \nu_{\rm ii} = \nu_{\rm i}, \tag{59}$$

identical to the momentum relaxation rates as noted above.

The ratios between the four thermalisation or heat exchange rates can be related to those between the four collision or momentum exchange rates as follows,

$$\nu_{\rm ie}^{\varepsilon} : \nu_{\rm ii}^{\varepsilon} : \nu_{\rm ee}^{\varepsilon} : \nu_{\rm ei}^{\varepsilon} \approx 2\nu_{\rm ie} : \nu_{\rm ii} : \nu_{\rm ee} : 2\left(\frac{m_{\rm e}}{m_{\rm i}}\right)\nu_{\rm ei}. \tag{60}$$

Note that the only significant difference in the relative magnitude of the heat and momentum exchange rates occurs for electrons colliding with ions, which differ by twice the mass ratio. Combining (60) with (43) we find,

$$\nu_{\rm ie}^{\varepsilon} : \nu_{\rm ii}^{\varepsilon} : \nu_{\rm ee}^{\varepsilon} : \nu_{\rm ei}^{\varepsilon} \approx Z : \left(\frac{m_{\rm i}}{m_{\rm e}}\right)^{1/2}\left(\frac{T_{\rm e}}{T_{\rm i}}\right)^{3/2}\frac{Z^2}{2^{3/2}} : \left(\frac{m_{\rm i}}{m_{\rm e}}\right)\frac{1}{2^{3/2}Z} : 1, \tag{61}$$

which expresses the well known result that electrons thermalise, or approach local thermodynamic equilibrium, much faster than ions, and that ions thermalise much faster than the rate of thermal equipartition between the ions and electrons. The rates of these three process differ roughly by the square root of the ion-to-electron mass ratio.

4. Comparison of collisional expressions in the literature

The rates of momentum and heat exchange due to Coulomb collisions for Maxwellian test and field particles are given by numerous sources in the plasma physics and magnetically confined fusion literature. In this section, we compare the expressions found in roughly twenty popular texts with those summarised in sections 2 and 3. The results are shown in table 2, which lists collisional rates reported by [8–11, 13–20] normalised by the expressions given here, i.e. equations (36), (39), (40), (41), (42), (44), (45) and (56). A dash in the table indicates that the quantity does not appear in the reference, while an asterix means that the quantity is only given for a pure, hydrogenic plasma ($Z = 1$). Note the frequent appearance of the factor of $2^{1/2}$ which reflects the neglect of the reduced mass in many definitions. This omission is then remedied by defining an additional ion collision rate, $\nu_{\rm i}$, which differs from $\nu_{\rm ii}$, precisely by a factor of $2^{1/2}$ [REFS]. The additional factors include the square root of the mass ratio, $a = (m_{\rm i}/m_{\rm e})^{1/2}$ and the 3/2 power of the temperature ratio, $b = (T_{\rm e}/T_{\rm i})^{3/2}$.

It is reassuring that several references, namely [8–11,13–16], report identical expressions (although not all sources derive all expressions). The expressions found in these sources can thus be used interchangeably, aside from a trivial conversion between Gaussian and SI units and taking care of the definition of the thermal velocities which differs between the various sources.

However, the reader should be aware of certain shortcomings in the definitions found in the NRL plasma formulary [15]: (i) the absence of an explicit charge scaling in $\tau_{\rm e}$ and $\tau_{\rm i}$, (ii) the violation of the usual convention $\nu = \tau^{-1}$, in the relation between the ion collision time and frequency ($\tau_{\rm i} = 2^{1/2}\nu_{\rm i}^{-1}$ for the expressions given in [15]), (iii) the absence of an explicit expression for $\nu_{ss'}$, and (iv) the suggestion that $\nu_{ss'}$ may be estimated by evaluating

Table 2. The ratio of collision rates defined in selected sources with those derived in [9] and compiled in sections 2 and (3). A dash means that the quantity does not appear in the reference, while an asterix means the quantity is only derived for $Z = 1$. The two constants appearing in the table are defined as $a = (m_\mathrm{i}/m_\mathrm{e})^{1/2}$ and $b = (T_\mathrm{e}/T_\mathrm{i})^{3/2}$.

Reference	$\nu_{ss'}/(35)$	$\nu_\mathrm{ei}/(39)$	$\nu_\mathrm{ie}/(40)$	$\nu_\mathrm{ee}/(41)$	$\nu_\mathrm{ii}/(42)$	$\nu_\mathrm{e}/(44)$	$\nu_\mathrm{i}/(45)$	$\nu^{\varepsilon}_\mathrm{ei}/(56)$
[9]	1	1	1	1	1	-	-	1
[10]	1	1	1	1	1	1	-	1
[11–14]	-	-	-	-	-	1	1	1
[15]	-	-	-	-	-	1^*	1^*	1
[16]	-	1	-	-	-	1	1	1
[17]	-	1	ab	$2^{1/2}$	$2^{1/2}$	Z^{-1}	1	1
[18]	-	1	2	-	$2^{1/2}$	Z^{-1}	$2^{1/2}$	1
[19]	-	1	-	$2^{1/2}$	$2^{1/2}$	-	-	Z^{-1}
[20]	-	-	-	-	-	1	$2^{1/2}$	1

the velocity dependent expressions, (1)–(4) with $\mathrm{v}_s = \mathrm{v}_{Ts}$, which, as is clear from table 1, leads to errors of order unity.

The expressions found in [17–20], differ from the definitions given here in several instances: ν_ie [17, 18], ν_ee [17, 19], ν_e for $Z \neq 1$ [17, 18], ν_ii [17–19] and ν_i [18, 20]. The difference by a factor $2^{1/2}$ can be traced to the replacement of the reduced mass by the test particle mass in (34). The discrepancy in ν_i found in reference [18] is of particular concern, since the parallel and perpendicular viscosities and ion heat conductivities given in that reference differ from those derived by Braginskii [11] by a factor of $2^{1/2}$.

The remaining sources [21–25], either do not explicitly state the collisional rates for Maxwellian test and field particles, or invoke various approximations in the course of the derivation. For this reason they are omitted from table 2 and are not recommended as quantitative references for collisional expressions, although they are very useful for more advanced material, e.g. collisional isotropisation rates, the effect of strong magnetic fields, etc. Therefore, it is important to consider the exact treatment of collisional terms in these references, which we consider briefly below.

In the monograph by Krall and Trivelpiece [21], the slowing down, deflection and energy loss rates are evaluated, but are not integrated for Maxwellian field particles. Instead the two-Maxwellian result is introduced via an effective collisional cross section which is referred to [8]. Similarly, the article by Sanderson [22] and a recent textbook by Boyd and Sanderson [23], contain a clear derivation of the slowing down, deflection and energy loss rates, (1), (2) and (4), but do not integrate these expressions for Maxwellian field particles. Instead, the authors approximate $\nu_{ss'}$ and $\nu^{\varepsilon}_{ss'}$ by evaluating the velocity dependent expressions for $\mathrm{v}_s = \mathrm{v}_{Ts}$, which introduces errors of order unity in the expressions for both momentum and energy exchange rates, tables 1 and 3. Finally, the two texts by Ichimaru [24, 25], which focus on a statistical description of plasma behavior, derive the correct electron-ion collision rate, but do so indirectly in the course of deriving the classical diffusion coefficient.

Table 3. The ratio of thermalisation rates defined in selected sources with those derived in [9] and compiled in section 3.

Reference	$\nu^{\varepsilon}_{ss'}$/(35)	$\nu^{\varepsilon}_{\rm ei}$/(39)	$\nu^{\varepsilon}_{\rm ie}$/(40)	$\nu^{\varepsilon}_{\rm ee}$/(41)	$\nu^{\varepsilon}_{\rm ii}$/(42)
[8–10]	1	1	1	1	1
[18]	-	1	-	$2^{1/2}$	$2^{1/2}$
[22, 23]	-	1	Z^{-1}	$2^{3/2}Z$	$2^{3/2}/bZ^2$

5. Conclusions

Having reviewed the expressions for binary Coulomb collision rates for Maxwellian test and field particles, we conclude that while the topical literature is generally consistent, there exist some discrepancies between the cited expressions, typically by factors of order unity. The most comprehensive treatment of the topic appears to be the review article by Hinton [9]. This source is recommended as the ultimate reference for all collisional rates, whether velocity-dependent or integrated over Maxwellian test particles, and has been adopted as the standard to which other sources can be benchmarked. The scaling factors which should be adopted when using expressions from other sources are summarised in tables 2 and 3. Special care should be taken when using the *NRL Plasma Formulary* [15], which contains several inconsistencies in the definitions of τ_s, ν_s and $\nu_{ss'}$, as outlined in section 4.

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References

[1] Spitzer L 1940 *Month. Not. Royal Astro. Soc.* **100** 396
[2] Chandrasekhar S 1941 *Astrophys. J.* **93** 285
[3] Chandrasekhar S 1942 *Principles of Stellar Dynamics* (U. Chicago Press, Chicago)
[4] Rosenbluth M N, MacDonald W M, Judd D L 1957 *Phys. Rev.* **107** 1
[5] Trubnikov B A 1958 *Sov. Phys. Doklady* **3** 136
[6] Trubnikov B A 1965 *Rev. Plasma Phys.* **1** 105
[7] Spitzer L and Härm R 1953 *Phys. Rev.* **89** 977
[8] Spitzer L 1956 *The Physics of fully ionized gases* (Interscience, New York)
[9] Hinton F L 1983 *Collisional transport in plasma* pp. 147–197 in Rosenbluth M N and Sagdeev R Z (Eds.) 1983 *Handbook of Plasma Physics* (North-Holland, New York)
[10] Hazeltine R D and Waelbroeck F L 2004 *The Framework of Plasma Physics* (Westview Press, Boulder, CO)
[11] Braginskii S I 1965 *Rev. Plasma Phys.* **1** 205
[12] Sivukhin D V 1966 *Rev. Plasma Phys.* **4** 93

[13] Wesson J 1997 *Tokamaks* (Oxford University Press, Oxford, UK)

[14] Kurlsrud R M 1983 *MHD description of plasma* 147–197 in Rosenbluth M N and Sagdeev R Z (Eds.) 1983 *Handbook of Plasma Physics* (North-Holland, New York)

[15] Huba J D 2006 *NRL Plasma Formulary* (Naval Research Laboratory, Washington, DC)

[16] Hazeltine R D and Meiss J D 1992 *Plasma Confinement* (Addison–Wesley, New York)

[17] Helander P and Sigmar D J 2002 *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge)

[18] Miyamoto K 1987 *Plasma Physics for Nuclear Fusion* (Cambridge, Massachusetts: MIT Press)

[19] Goldston R J and Rutherford P H 1995 *Introduction to Plasma Physics* (IoP Press, Bristol, UK)

[20] Balescu R 1988 *Transport Processes in Plasmas, 1. Classical transport theory* (North-Holland, Amsterdam)

[21] Krall and Trivelpiece 1976 *Principles of Plasma Physics*, (Interscience, New York)

[22] Sanderson J J 1981 *Kinetic Theory* in Gill R D ed. *Plasma Physics and Nuclear Fusion Research* (Academic Press, London)

[23] Boyd T J M and Sanderson J J 2003 *The Physics of Plasmas*, (Cambridge University Press, Cambridge)

[24] Ichimaru S 1992 *Statistical Plasma Physics, Volume I*, (Addison–Wesley, New York)

[25] Ichimaru S 1973 *Basic Principles of Plasma Physics: A statistical Approach*, (W. A. Benjamin, Inc., Reading, Mass.)