A Crude Model to Study the Interplay of Radio Frequency Waves and the Density Close to Launchers and Metallic Surfaces

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A crude model to study the interplay of radio frequency waves and the density close to launchers and metallic surfaces

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The interplay between radio frequency (RF) waves and the density is discussed adopting the general framework of a 2-time-scale multi-fluid treatment allowing to separate the dynamics on the RF time scale from that on the time scale on which macroscopic density and flows vary as a result of the presence of electromagnetic or electrostatic fields. The focus is on RF sheaths formed close to metallic walls but the adopted equations offer a framework that can be applied more widely. Fast time scale dynamics impact on the slow time scale via quasilinear terms (the Ponderomotive force for the case of the equation of motion). Opposite to what is traditionally done, electrons and ions are treated on the same footing. Also, both fast and slow waves are retained in the wave description. Although this work is meant as a subtopic of a larger study - the physics at hand is of a 2D nature while this paper limits itself at 1D - a few tentative examples are presented.

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I. INTRODUCTION

The present paper discusses the interplay of radio frequency waves and the plasma close to wave launchers and metallic surfaces. It aims at providing a crude model for wave induced density modification. Although the physics at hand is multidimensional, the practical examples given are 1-dimensional. For the specific case of the sheath formed near metal surfaces, this reduction to a single dominant direction can be easily be justified when realizing that charge separation is only significant on the Debye length scale; the variations in the 2 directions parallel to the wall are expected to occur on scalelengths much larger than the Debye length. But the here proposed expressions can also be used to describe the wave-plasma interplay more macroscopically in the antenna-near-field region, be it that extra ingredients (density 'source' terms due to imperfect screening at the last closed flux surface and spatial diffusion to name just 2 when thinking of the wave induced density modification close to wave launchers) are needed in that case and that accounting for multiple dimensions then becomes essential. The sheath dynamics is due to the charge separation arising from the violation of the charge neutrality caused by the differing electron and ions dynamics on account of their very different mass; as electrons are much more mobile, metallic walls tend to charge negatively.

In view of the very different time scales involved in RF wave physics, it makes sense to split the task of modelling the wave-plasma interaction into 2 subproblems: one describing the fast time scale dynamics (and in which the slow time scale variables can simply be assumed to be constant as a function of time), and one describing the slow time scale dynamics (for which the fast dynamics only intervene through second order corrections averaged over the relevant fast dynamics period or periods). All quantities are assumed to be the sum of a slowly varying term, topped up by a small, rapidly varying perturbation. This justifies linearizing the equations and obtaining a description for the deviations away from the slow time scale dynamics. Moreover, assuming transient behavior is of no concern, the fast time scale physics can be studied as a driven response to an oscillating drive at a prescribed frequency \( \omega \) i.e. \( \partial / \partial t \to -i\omega \). When relevant other fast scale variations (in this particular case the time dependence following from the Larmor gyrations the strong magnetic field imposes on the charged particles) equally need to be removed by a proper averaging when seeking to isolate the 'pure' slow time scale response. Following this approach, the dynamics
can be described by separate but coupled sets of equations, one set for each of the 2 time scales. The fast time scale description requires slow time scale quantities as input. The slow time scale description feels the influence of the fast time scale via quasilinear terms: Although the period-averaged perturbed quantities vanish, quadratic contributions do not and hence their time average adds a finite contribution to the slow time scale equations.

The pioneering work of Myra and D’Ippolito on the subject of the RF sheath has been - and still is - instrumental (see e.g.\textsuperscript{2-10}). They realised that the sheath dynamics needs to be accounted for when studying radio frequency wave excitation and damping in tokamak plasmas but that the very short range over which the sheath extends makes it impossible to encorporate sheath details into wave codes without dramatically increasing the memory requirements to solve the relevant wave equation. They proposed an elegant way out: Replacing the metallic boundary conditions by sheath boundary conditions. These conditions account for the presence of the microscopic sheath layer but do not require to handle the details in a macroscopic wave code. This allows to extend the usual near-antenna-field modelling to include some key aspects of the impact of the sheath, treating it as a related but separate problem. Kohno adopted the sheath boundary condition proposed by Myra and D’Ippolito to study the wave dynamics in the neighborhood of the antenna\textsuperscript{11,12}. Similarly, Colas\textsuperscript{13} and Jacquot\textsuperscript{14} solved an equation for the slow time scale voltage and couple it to the slow wave solution on the fast time scale; the fast wave is excluded in their description and the metallic walls are assumed to be electrically grounded. Recent work by Lu is ongoing to add further wave physics - including the fast wave and hence accounting for all waves types admitted by the cold plasma dielectric tensor - to the model proposed by Jacquot and Colas\textsuperscript{15}.

On the road to formulating their sheath boundary condition, Myra and D’Ippolito made a number of assumptions. The displacement current is assumed to be simply proportional to the electric field (i.e. the dielectric tensor is replaced by a scalar sheath permittivity), which is assumed to be electrostatic. Adopting the Child-Langmuir law, the sheath width is estimated and the partial derivative in the direction normal to the metallic wall is approximated by a finite difference, implicitly assuming the potential is linear. Finally, the tangental gradient of the obtained quantity is evaluated to yield the 2 finite tangential electric field components replacing the 2 boundary conditions $\vec{E}_{tangential} = 0$, assuming there is no charge on the interface between the sheath and the ‘main’ plasma. The sheath potential
is assumed to be related to the RF sheath potential by a rectification constant, and the radio
frequency voltage $V_{RF}$ is taken to be the integral of the parallel electric field along the field
line. Adopting a 1D parallel plate capacitor model, Riyopoulos\textsuperscript{16} and Myra\textsuperscript{2}
found a value for the rectification constant; it typically takes values around 0.5 but various parameters
(e.g. the density, the orientation of the ion flow and that of the magnetic field) have been
shown to have a non-negligible impact on it. For various reasons it seems worthwhile to
alleviate assumptions made to arrive at the sheath boundary condition, and to check if the
resulting framework can be applied in a wider context.

The present paper aims at offering a simple, consistent, first-principles framework for
describing density modifications brought about by radio frequency waves, and at exploring
this framework to describe the dynamics in the RF sheath formed close to metallic walls in
tokamak environment in presence both of a strong confining magnetic field and RF power.
In doing so, it illustrates that it is unlikely that the complex dynamics in a sheath can
ultimately be captured in a simple boundary condition. This paper is structured as follows:
First, a number of aspects relevant for explaining the later adopted philosophy are briefly
discussed. After that, the relevant set of equations is sketched and/or derived. Different
subsections are devoted to the various equations. After that, the actual 1D version of
the adopted set is sketched and the adopted method of solution is discussed. Next, some
examples are given. Finally, conclusions are drawn and a brief discussion sketches the way
ahead.

Up to few exceptions, the ingredients used in this paper are 'classical' and only require
brief refreshment; in most cases it will suffice to provide references where the interested
reader can find more details. The fact that the building blocks are 'classical' does not
guarantee they are not limited ... The model that will be developed here is a fluid model
and hence - by definition - it excludes kinetic effects. In particular it overlooks the fact
that the velocity distribution function of the particles does not need to have a gaussian
shape at the modest densities close to the antennas and were the time spent by a particle
in the modeled region is modest; the implicitly made assumption that particles sample a
given location many times - as is the case for a particle on a closed magnetic surface in a
tokamak - is not justified when modelling plasma behind the last closed flux surface and in
particular inside the antenna box. As a result, the routinely applied averaging underlying
the splitting of 2 time scales may become questionable for some applications. Moreover,
quasilinear theory is applied, which - given the magnitude of the electric fields carrying MegaWatts of power into the tokamak - hinges on the assumption that the net effect of a rapid succession of events on a fast time scale can sensefully be captured by a slow time scale model in which the effect of the fast time scale shows up as a small, finite average over the oscillation period (or periods) of the fast time scale. It suffices to check the work of Dirickx1 on transport equations to realize that strong gradients necessitate the re-development of the whole drift ordering theory of the interaction of charged particles in the presence of electric and magnetic fields, a concept far beyond the scope of the present paper. In the particular case of the sheath formed close to metallic walls where charge separation occurs, the steepness of the density gradient is usually insufficient to invalidate the assumption that the Larmor radius is small w.r.t. the relevant scale length (Debye length) for electrons but potentially poses an issue for the ions. Effects of turbulence are omitted altogether, and the fact that at very modest energies the strength of a typical tokamak magnetic field may be insufficient to ensure that magnetic confinement even survives in the presence of strong electric fields is not addressed either. In view of its many weaknesses, one may wonder if it makes sense to even attempt to adopt the framework of a fluid theory and the quasilinear approximation to capture near-field physics. The authors’ approach is pragmatical: In view of the lack of a fully developed theory incorporating steep gradients, kinetic effects and an integro-differential description of the dielectric response, it seems worthwhile to assess the potential of a simplified theory to at least present a qualitative, consistent picture of the dynamics that can be used to help understand experimental findings. Lacking certain ingredients (the 2D nature of the physics and the inclusion of transport being trivial ones), it is clear that quantitative agreement cannot be hoped for and requires further work. On the other hand, the model presented here combines many of the contributions identified as important in the literature in various limits, and does so yielding a model that is not overly time consuming. It proposes a crude framework for describing wave-plasma interaction in presence of a strong static magnetic field with arbitrary magnetic field direction, and treats electrons and ions on a common footing.
II. PRELIMINARY REFLECTIONS ON ASPECTS OF THE ADOPTED PHILOSOPHY

A vast amount of literature exists on sheath dynamics; a fairly extensive list was provided in\textsuperscript{17,18} and interested readers can consult it there. A few key points addressed in that paper are recaptured here:

Godyak developed a self-consistent dynamical model for RF sheaths with frequencies between the ion and electron plasma frequencies, but omitted the presence of a confining magnetic field\textsuperscript{19,20}, an essential ingredient for applications in a tokamak. Godyak’s results converge to known limits in the high voltage approximation but for voltages of only tens and hundreds of volts they disagree while being in good agreement with the experiment. For sufficiently small volumes, Particle-in-cell (PIC) simulations have allowed to grasp the sheath physics (see e.g.\textsuperscript{21,22}): PIC simulations have successfully reproduced experimental findings for argon capacitive RF discharges. In magnetic confinement machines the relevant volumes are much larger and PIC modelling encounters the problem of tracing a sufficiently large number of particles to make the description statistically relevant. Nonetheless, PIC methods may be instrumental in helping to understand the RF induced sheaths as they have the clear advantage of being able to include physics processes from first principles.

Opposite to Godyak’s work, Stangeby addressed the sheath problem for the case ICRH waves are absent but a confining magnetic field is present\textsuperscript{23,24}. It does not come as a surprise that the magnetic field strength and its direction w.r.t. the metal wall in front of which the sheath forms impacts significantly on the results: While in a sheath the dynamics normal to a metallic surface is characterized by scalelengths much shorter than that parallel to it, the particle motion parallel to a strong confining field is much more free than perpendicular to the field lines. One thus does expect that aligning the field with the wall is a limit that requires special attention.

modelling RF sheaths near metallic walls requires an approach that accounts both for the presence of the strong confining magnetic field and the rapidly varying electromagnetic field. Although still simplified in various respects, the present paper aims to offer such a model. It is a continuation of the efforts reported in earlier publications\textsuperscript{17,18} which, respectively, addressed the fast time scale wave dynamics in the sheath for a prescribed density profile, and the wave induced density depletion in the antenna region without accounting for the
role the waves play in setting up the density profile. The present paper adds the role of the RF fields but is currently restricted to 1D applications in spite of the fact that the dynamics is multidimensional in nature.

Simple sheath models assume that ions are immobile while the electrons have a Boltzmann distribution. This distribution is the solution of a simplified version of the equation of motion in which the density responds to the local potential to ensure the pressure change is balanced. In practice the ions are not immobile, however. Sputtering and hot spots would not be an issue if the approximate description of immobile ions would be close to reality...

When ions and electrons respond differently to reigning forces, charge separation occurs and the electrostatic potential change across the sheath is no longer linear. Assuming the total energy is constant and that only the electrons are creating a charge separation allows to relate the electron velocity to the value of the potential

$$mv^2/2 - e\Phi = 0 \rightarrow v = (2e\Phi/m)^{1/2}$$

and hence to relate the charge density $\rho_e$ in the right hand side of Poisson’s equation with the potential $\Phi$ via the (electron) current $J_e = \rho_e v$. This allows to rewrite the 1-D Poisson equation as

$$d^2\Phi/dx^2 = \xi/\Phi^{1/2}$$

where $\xi = -J_e/\epsilon_o(m_e/2e)^{1/2}$. Multiplying both sides by $d\Phi/dx$ this equation can readily be integrated to yield the Child-Langmuir law $\Phi = (3/2)^{4/3}\xi^{2/3}x^{4/3}$. The latter shows that the sheath width scales with the Debye length and with $(e\Phi/T)^{3/4}$, where $T$ is the temperature. This approximation breaks down when a more refined model is used in which ions are not immobile and in which the energy balance is computed accounting for other forces than the pressure and the electrostatic potential. But it allows to get a first guess of the dynamics involved.

The ion velocity at the interface of the sheath with the main plasma is commonly assumed to be equal to or larger than the local sound velocity (“Bohm criterium”; see e.g.26). Starting from the assumption - based on a simplified version of the continuity equation - that the ion density is inversely proportional to the velocity ($N_i v_i = \alpha t$), considering the total energy to be constant and retaining the leading order term in the Taylor series expansion of the electron density assumed to be of the Boltzmann form $N_e \propto exp[e\Phi/kT_e]$ when solving the Poisson equation yields the Bohm criterium $v_i \geq (kT_e/m_i)^{1/2}$. The same model also allows
to estimate the potential drop at the wall as compared to the quasi-neutral region to be of order $3kT_e/e$, in which the factor 3 comes from the difference between the electron and ion mass. Callen demonstrates that an actual sheath is formed when the Bohm criterium is satisfied but equally shows that ion velocities can very well be smaller, in which case an actual sheath does not form but an oscillatory rather than exponential behavior is observed close to the metallic wall instead\textsuperscript{26}. On the other hand, Zhang pointed out that ion streaming instabilities can occur in plasmas with multiple ion species\textsuperscript{27}, highlighting the fact that properly capturing the slow time scale dynamics is likely impossible when adopting too simple models that do not represent the interplay between particles with different charges and concentrations. Ji discussed the role of the magnetic field on the Bohm criterium and found that the electric field at the plasma-sheath boundary has an important role in the sheath criterion\textsuperscript{28}. Yankun also looked into the Bohm criterion in presence of a - small - magnetic field\textsuperscript{29}. That work confirms that both the strength and orientation of the magnetic field affect the ion speed, and thereby the density profile that is set up.

Interpreting the potential appearing in the Boltzmann expression for the electron density as the sum of an electrostatic and a driven, rapidly varying part allows to have a first, crude guess on how RF waves impact on the density. Such an approach is standard practice when looking at the RF sheath as an electrical circuit (see e.g.\textsuperscript{30}). The mean value of the density averaged over a period $T_{RF}$ of the driver can be computed to be

$$\frac{1}{T_{RF}} \int_0^{T_{RF}} N_{ref} \exp \left( \frac{e\Phi}{kT} \right) dt = \frac{N_{ref}}{T_{RF}} \int_0^{T_{RF}} \exp \left( \frac{e\Phi}{kT} \right) + \frac{e\Phi_{RF}}{kT} \cos \omega t \right) dt$$

$$= \left[ N_{ref} \exp \left( \frac{e\Phi}{kT} \right) \right]_0^{T_{RF}} \left( \frac{e\Phi_{RF}}{kT} \right)$$

in which the first factor right of the rightmost equality sign represents the density in absence of RF waves while the second factor involving the modified Bessel function of order zero reveals that the net effect of the RF oscillation is a \textit{reduction} of the net electron density. The larger the amplitude of the perturbation, the more pronounced the density decrease. The present paper deviates in spirit from the description in which the total potential is considered to be the sum of a fixed or slow varying contribution and a contribution that oscillates at the frequency of the driver. The reason for this comes from the adopted philosophy: On the slow time scale $\partial/\partial t$ is assumed to be small enough to justify it being dropped altogether. Hence, Faraday’s law states that the curl of the electric field is zero and so that the field is
the gradient of a potential. By definition, that potential does not depend on time. On the fast time scale there is no immediate reason to assume the electric field can be represented as a potential, and hence there is no immediate justification on this time scale either to simply write the driven response as due to a potential, in particular when studying - as is the case in the present paper - wave dynamics in the ion cyclotron domain of frequencies where launching the - electromagnetic - fast wave adopting poloidally oriented straps is routinely done. That does not exclude that going through the full derivation cannot yield the result that the dominant effect is due to a wave that is essentially electrostatic so that models can be re-simplified a posteriori ...

One aspect that has - purposely - been left out of the present description is the impact of kinetic corrections. It has been shown in various computations\textsuperscript{31,32} that such corrections do matter when looking into the more detailed description of the wave-particle interaction. On the other hand, such kinetic computations show that the key deviations of the distribution function from a Maxwellian is that zero order flows need to be accounted for. In a somewhat crude way, the notion of temperature linked to the spread of the distribution around the average flow velocity in velocity space survives, be it that kinetic computations show that deformations away from a Gaussian spread are observed. Kinetic models (see e.g.\textsuperscript{33–35}) automatically incorporate zero order flow and the deviations of the distribution away from the average flow. This type of modelling e.g. showed that the Bohm criterium does not uniformly need to be satisfied in the sheath dynamics context and that the sheath is not limited to a few Debye lengths but may be substantially larger\textsuperscript{31,33}, stating that not only the close vicinity of the metallic wall should ultimately be examined but a more macroscopic view is possibly needed. At the price of not being able to account for the impact of asymmetries (deviations away from Gaussian distributions around average flow velocities in velocity space) identified in kinetic models, the model adopted in the present paper is a multifluid model, every species being characterised by a local density and a temperature (representative for the thermal spread around the fluid flow). Moreover, it is assumed that the latter is constant in the domain of interest. When studying sputtering and the formation of hot spots caused by ions bombarding the wall, such temperature variations are not necessarily unimportant. At this stage, the authors, however, assume their impact on the macroscopic density is sufficiently modest to allow cutting down the required model to the simplest possible multi-fluid description.
III. THE RELEVANT SET OF EQUATIONS

To model the interplay of a plasma and electromagnetic waves, 3 types of equations are a bare minimum to capture the essence:

- Solving Maxwell’s equations yields the electromagnetic field pattern arising for a given current density.

- Solving the equation of motion learns how the plasma momentum for each type of species is modified under the influence of the relevant forces, the Lorentz force in particular.

- Solving the continuity equation provides the density resulting from the flows resulting from the equation of motion for each population.

In the present paper, it is assumed that the temperature is constant. The above set needs to be supplemented with an energy evolution equation in case more detailed knowledge of the temperature is required. When describing the impact of radio frequency (RF) waves, 2 rather than 1 sets of such equations need to be studied since the total motion is the sum of a rapidly varying, direct response to the RF waves and a slowly varying response to the net forces still present after averaging over the fast time dynamics. In this section, the adopted equations are sketched and where needed derived.

A. Equation of motion (see e.g.\textsuperscript{17,36})

The starting point is the single particle equation of motion under the influence of the Lorentz force. Splitting the position and the velocity into a slow and a small but fast varying contribution ($\vec{x} = \vec{x}_o + \vec{\rho}_1$, $\vec{v} = \vec{v}_o + \vec{v}_1$) and neglecting higher than first order corrections of the fields, the slow flow is captured by solving the equation

$$m \frac{d\vec{v}_o}{dt} = q\vec{E}_o + q[\vec{v}_o \times \vec{B}_o] + \left\langle \frac{d\vec{\rho}_1}{dt} \times (\vec{\rho}_1.\nabla)\vec{B}_o \right\rangle + \vec{F}_{RF}$$  \hspace{1cm} (1)

in which $\vec{F}_{RF}$ is the net force due to the rapidly varying aspects of the motion, i.e.

$$\vec{F}_{RF} = \left\langle q[\vec{v}_1.\nabla\vec{E}_1 + \vec{v}_1 \times \vec{B}_1 + \frac{d\vec{\rho}_1}{dt} \times (\vec{\rho}_1.\nabla)\vec{B}_o] \right\rangle$$  \hspace{1cm} (2)
and where \(< ... >\) denotes the smoothing time average; for convenience the cyclotron and driven motions are split into the cyclotron rotation part (denoted with a subscript '\(\Omega\)') and a driven part. By definition, the driven oscillation (denoted with the subscript '1') satisfies

\[
\vec{v}_1 = -i\omega \vec{\rho}_1 \quad (3)
\]

\[-i\omega m \vec{v}_1 = q[\vec{E}_1 + \vec{v}_1 \times \vec{B}_0]\]

in which \(\omega\) is the driver frequency and a driven response \(\propto \exp[-i\omega t]\) was assumed. \(\vec{B}_0\) and \(\vec{E}_0\) are the static magnetic and electric fields. The \(\vec{F}_{RF}\) term is known as the Ponderomotive force (see e.g.\(^{37}\) for the usual derivation in absence of a magnetic field). Assuming the static magnetic field gradient is negligibly small and adopting complex notation so that the time average of a quadratic driven quantity over a period is \(< AB > = \text{Re}[A^*B]/2\), the corresponding acceleration can be written as

\[
\vec{a}_{Pond} = -\nabla \Theta \quad (4)
\]

where

\[
\Theta = \frac{1}{4} \text{Re} \left[ \vec{v}_1.\vec{v}_1^* + \frac{i\vec{\Omega}}{\omega} \vec{v}_1 \times \vec{v}_1^* \right] \quad (5)
\]

and in which \(\vec{\Omega} = q\vec{B}_o/m\). Introducing the notation \(\vec{\epsilon} = q\vec{E}_1/m\), \(\Theta\) can readily be written in terms of the RF electric field:

\[
\Theta = \frac{1}{4} \left[ \frac{1}{\omega^2 - \Omega^2} \left| \vec{\epsilon} \right|^2 - \frac{1}{\omega^2} \left| \vec{\epsilon} \cdot \vec{\Omega} \right|^2 + \frac{i}{\omega} \vec{\Omega} \cdot \vec{\epsilon} \times \vec{\epsilon} \right] = \frac{1}{4} \left[ \frac{1}{\omega^2 - \Omega^2} \left| \vec{\epsilon} \right|^2 - \frac{1}{\omega} \left| \vec{\epsilon} \cdot \vec{\Omega} \right|^2 - \frac{2\Omega}{\omega} \text{Im}[\epsilon^*_{\perp}\epsilon_{\perp,2}] \right]. \quad (6)
\]

To obtain the above, the expression

\[
\vec{v}_1 = \frac{1}{\Omega^2 - \omega^2} \left[ -i\omega \vec{\epsilon} + \vec{\epsilon} \times \vec{\Omega} + \frac{i\vec{\Omega}}{\omega} \vec{\epsilon} \times \vec{\Omega} \right] \quad (7)
\]

is useful. With the obtained expressions and simplifications, the equation of motion is now compactly

\[
\frac{d\vec{v}_o}{dt} = -\frac{q\nabla \Phi}{m} - \nabla \Theta + \Omega \vec{v}_o \times \vec{\epsilon}/ \quad (8)
\]

in which the notation \(\vec{E}_o = -\nabla \Phi\) was introduced to reflect the fact that Faraday’s law requires the electric field \(\vec{E}_o\) to be the gradient of a function when \(\partial/\partial t = 0\). Klima extends
the result found for single particles to the case of a flow. Provided the averaging is done in Lagrangian coordinates rather than with respect to a fixed point in space, one merely needs to add the pressure term \(-\nabla P = -\nabla kTN_o\). Assuming the temperature is constant one then immediately finds the generalised equation of motion

\[
\frac{d\vec{v}_o}{dt} = -v_i^2 \frac{\nabla N_o}{N_o} - \frac{q\nabla \Phi}{m} - \nabla \Theta + \Omega \vec{v}_o \times \vec{e}_{//} \tag{9}
\]

in which \(N_o\) is the density and \(v_i^2 = kT/m\) is the square of the thermal velocity; \(T\) is the temperature. Whereas the perpendicular component of this equation still forces rapid oscillations through the Lorentz term \(\Omega \vec{v}_o \times \vec{e}_{//}\), the parallel component does not. For a given static acceleration \(\vec{a} = -v_i^2 [\nabla N_o]/N_o - q\nabla \Phi/m - \nabla \Theta\), the solution of the perpendicular part of the equation of motion is

\[
v_+ = v_{+0} \exp[-i\Omega t] + \frac{a_+}{i\Omega} \tag{10}
\]

\[
v_- = v_{-0} \exp[+i\Omega t] + \frac{a_-}{-i\Omega}
\]

(where \(v_\pm = v_{\perp,1} \pm iv_{\perp,2}\)) which consists of a rapidly varying contribution oscillating at the cyclotron frequency, and a slow drift term across the magnetic field lines of the usual form

\[
\vec{v}_{\text{drift},\perp} = \frac{1}{qB_o^2} \vec{E} \times \vec{B}_o = \frac{1}{\Omega^2} [\vec{a} \times \vec{e}_{//}] = \frac{1}{\Omega^2} [\vec{e}_{\perp,1} a_{\perp,2} - \vec{e}_{\perp,2} a_{\perp,1}]. \tag{11}
\]

Aside from the diamagnetic drift due to the \(\nabla P = kT\nabla N_o\) force and the \(\vec{E}_o \times \vec{B}_o\) drift due to the electrostatic field \(\vec{E}_o = -\nabla \Phi\), it consists of a Ponderomotive contribution due to the finite RF electric field. From this point in the paper onwards \(\vec{v}_o\) will represent the true slow time scale velocity i.e. the zero order velocity averaged over all fast time scale phenomena so not only over the driver period but also that over the cyclotron period \((\vec{v}_o \rightarrow 1/[2\pi] \int_0^{2\pi} \vec{v}_o d\phi\) in which \(\phi\) is the gyro-angle). Note that the perpendicular drift is divergence-free provided \(\Omega\) is constant, and that it conserves the perpendicular energy \((\vec{a}_\perp \cdot \vec{v}_{\text{drift,}\perp} = 0)\) i.e. it is a force-free flow; by definition the parallel acceleration is not affected by the perpendicular drift (the parallel flow is discussed later).

A key point to the approach adopted is that Eq. 9 is solved by hand fully including the time dependence upon considering the forces aside from the Lorentz force due to the confining magnetic field to be static. The Lorentz force term explicitly contains the velocity \(\vec{v}_o\) and hence cannot off-hand be considered to be time-independent. The full equation is
solved by first solving the equation in absence of \( \vec{a} \) and assuming \( \vec{B}_o \) to be constant, and then by variation of constants to introduce the effect of the finite \( \vec{a} \). In the latter the effect of a slightly inhomogeneous magnetic field could be introduced by a supplementary \( m/[qB_o^2] [v_\perp^2 + v_\parallel^2/2] \vec{B}_o \times \nabla B_o \) velocity drift term: on the scale length of the sheath - the main region of interest in this paper - the changes of \( \vec{B}_o \) in tokamak geometry are modest both in amplitude and direction. Note that the obtained perpendicular drift satisfies \( d\vec{v}_o/dt \) in an elegant way: it is time independent (hence \( \partial \vec{v}_o/\partial t = 0 \)) and it is homogeneous (hence \( \nabla \vec{v}_o = 0 \) and thus \( \vec{v}_o \cdot \nabla \vec{v}_o = 0 \)). As a result, the simplified equation \( \vec{a} + \vec{v}_o \times \vec{B}_o = 0 \) is satisfied.

The fact that the acceleration \( \vec{a} \) is static does not mean that its effect is small: In the sheath, the gradient of the density is steep and hence the drift perpendicular to the magnetic field and - depending on the \( \vec{B}_o \) orientation - parallel and/or perpendicular to the metallic wall is significant. On top of that, the equation for the parallel motion is nonlinear. As will be discussed later in this paper, the burden of finding the correct solution to the equation is put on the computer: rather than solving the non-linear equation directly, an iterative scheme is introduced which allows to write the nonlinear equation at hand as the asymptotic version of an iteration scheme of linear equations. Because drifts perpendicular to reigning gradients are created, the full dynamics cannot fully be captured adopting a 1D model.

The fast time scale effect was removed from the perpendicular dynamics by averaging over the cyclotron period, which yields a net drift. The magnetic field is incapable of accelerating particles along its field lines. Opposite to what was needed for the perpendicular motion, omitting the slow time variation altogether (\( \partial/\partial t \to 0 \)) is reasonable straightaway. Assuming that the confining magnetic field is strong so that the perpendicular drift velocity is modest - allowing it to be neglected as a first approximation (\( \vec{v}_o \cdot \nabla \approx v_\parallel \partial/\partial x_\parallel \); this approximation will be relaxed a few lines further) - the equation of motion for the parallel motion can be written

\[
v_\parallel \frac{\partial}{\partial x_\parallel} v_\parallel = -v_i^2 \frac{\partial \ln N_o}{\partial x_\parallel} - \frac{q}{m} \frac{\partial \Phi}{\partial x_\parallel} - \frac{\partial \Theta}{\partial x_\parallel}
\]

which can readily be integrated to yield

\[
N_o = N_{o, ref} exp[- \frac{1}{v_i^2} \left( \frac{v_\parallel^2}{2} + \frac{q \Phi}{m} + \Theta \right)].
\]
Note that - opposite to the perpendicular drift - the parallel motion has a finite convective derivative. Later in the text the variable $x$ parametrising the direction perpendicular to the wall will be used as independent variable rather than $x_{//}$. Hence the limiting case in which the confining magnetic field is parallel to the metallic wall cannot be described by the above equation or - more accurately - the density then is prescribed elsewhere and cannot be deduced from solving this equation. For the main focus of the paper - describing sheath dynamics - there is another and more fundamental reason for the adopted method not to be applicable: When the Larmor radius is of the same order as or larger than the sheath width, the gyro-averaging to remove the fast particle motion to arrive at a 'net' slow flow is a senseless procedure since particles with guiding centers too close to the wall then cannot execute their gyrorevolution and will hit the wall before completing a Larmor oscillation. For electrons and the strong fields in a tokamak, this does not pose a big problem, but for ions one needs to be more careful. This limiting case will implicitly be excluded from our modelling.

The above is a generalisation of the Boltzmann expression incorporating the impact of the parallel flow and of the RF Ponderomotive force aside the usual electrostatic potential. The terms $v_{//}^2/2$, $q\Phi/m$ and $\Theta$ play a similar role in the density variation and are all position-dependent. For a simple plane wave solution, $\vec{E} \propto \exp[ik_xx]$, the density change caused by the electric field scales as the derivative of $\Theta$: $\frac{d\ln N_0}{dx}_{RF} = 2Im[k_x]\Theta/v_2^2$. This readily shows that the effect of the waves on the density is much more pronounced when waves are evanescent than when waves are propagative. Furthermore, as $\Theta \propto |\vec{\epsilon}|^2$ the density modification scales with the square of the electric field. For both these reasons, wave-induced density depletion is expected to be much stronger close to wave launchers than away from them, and is most pronounced in low density sheath regions. Moreover, short wavelength branches have a stronger impact than long wavelength modes. In the practical examples shown later, it is assumed the integration interval reaches up to the point where the value of $\Theta$ no longer changes considerably. That does not mean there is no influence of the RF field on the density there. It just means there are no density modifications as a result of its presence at that location.

It is instructive to evaluate the Ponderomotive potential $\Phi_{Pond} = m\Theta/q$ for a typical case to get a sense of the magnitude of the term; recall that the electrostatic potential is of the order of a few times the electron temperature. In the ion cyclotron region of frequencies, the
electron cyclotron frequency is much larger than the driver frequency. Hence for electrons,

\[ \Phi_{\text{Pond},e} = -\frac{m_e \Theta_e}{e} \approx -1.11 \times 10^{-3} \left| \frac{E_{||}[V/m]}{f[MHz]} \right|^2. \]

For ions one gets

\[ \Phi_{\text{Pond},i} = \frac{m_i \Theta_i}{q_i} \approx 6.07 \times 10^{-7} \frac{Z_i}{\tilde{\alpha} A_i} \left| \frac{E[V/m]}{f[MHz]} \right|^2 \]

in which the factor \( \tilde{\alpha} \) is of order 1; it accounts for the difference in magnitude between \( \Omega \) and \( \omega \) and for the polarization. Electrons thus are primarily influenced by the parallel electric field while the ions feel the influence of the total field. In the expression of the density these terms appear as \( \Theta / v_i^2 = (q/kT) \Phi_{\text{Pond}} \). In JET, the A2 antenna is at major radius \( R_{\text{ant}} = 3.9 \) m in the equatorial plane. Central hydrogen minority heating in a deuterium plasma - the most commonly adopted ICRH heating scheme in JET - is typically done at \( B_o = 3.45 \) T and adopting \( f = 51 \) MHz. Neglecting the effect of the poloidal magnetic field this yields a magnetic field strength of \( 2.62 T \) at the antenna location and hence \( \Omega_D = 1.25 \times 10^8 \) rad/s and \( \Omega_e = -4.61 \times 10^{11} \) rad/s. For the majority ions, \( \Omega / \omega \approx 0.4 \) while for the electrons \( \omega \) can be neglected w.r.t. \( \Omega \). Consider an edge temperature of \( T_e = T_i = 5 \) eV. For the electrons \( (q/kT) \Phi_{\text{Pond}} = -1.11 \times 10^{-3} |E_{||}[V/m]/f[MHz]|^2/T_e \) and hence the density e-folding requires \( E_{||} \approx 3.4 \) keV/m. Figure 1 depicts the variation of the density along a magnetic field line as a result of the variation of the parallel velocity or of the potential for a deuterium plasma. As the electrostatic potential \( \Phi \) and the Ponderomotive potential \( m \Theta / q \) play a similar role, only \( \Phi \) was varied. Anticipating the fact that both the electron and ion flow velocity are of the order of the sound velocity \( c_s = (\gamma Z_i T/m_i)^{1/2} \) (Bohm criterium; for a mono-atomic gas \( \gamma = 1.4 - 1.6 \)), the parallel velocities are expressed in terms of \( c_s \) in the figure. Because the electron thermal velocity is much larger than the ion sound velocity, the electron density is hardly modified by changes of the parallel velocity. As the differing electron and ion mobilities cause the wall to charge negatively, the potential has been scanned only for negative values. Potential dips of up to \( 20 \) V (simple models show the sheath voltage is of order \( 3 T_e \)) cause the electron density depletion to be almost complete. Due to the dependence on the charge, the ion density potentially increases and does so by up to a factor 50 for fixed \( v_{||} \). An increase of the parallel velocity moderates this ion density increase. As will be shown later, the variation of the parallel velocity and that of the density are coupled. Hence the here depicted variation only indicates the trend and cannot not be used as a prediction for the actual density change. For the Ponderomotive potential
to be as important as the electrostatic potential in the exponent of the expression for the density requires $E_{\parallel} \approx 5kV/m$ for the electrons and $|E| \approx 300kV/m$ for the ions; a reference value 10V was used to obtain these values. Such values are perfectly feasible close to the launcher and in particular inside the antenna box of multi-MW launchers where voltages of tens of $kV$ are applied on straps that are only centimeters away from grounded metallic antenna boxes, but depending on the local $\vec{B}_o$ direction - they may not be representative very close to metallic surfaces. Remind, however, that it is not the actual value of $\Theta$ or $\Phi_{Pond}$ but its gradient that is important in the force balance. As the sheath region has a very limited extent but the density is non-negligible modified in it, steep gradients can occur and necessitate taking a closer look at the interplay of the various effects.

In general, the magnetic field is not strong enough for the drift terms in the parallel equation of motion to be neglected. The equation for $v_{\parallel}$ is then generalised to

$$v_{\parallel} \frac{\partial}{\partial x_{\parallel}} v_{\parallel} = -[\vec{v}_{drift,\perp}.\nabla v_{\parallel} + v_i^2 \frac{\partial \ln N_o}{\partial x_{\parallel}} + \frac{q}{m} \frac{\partial \Phi}{\partial x_{\parallel}} + \frac{\partial \Theta}{\partial x_{\parallel}}]. \quad (14)$$

This equation can no longer be integrated analytically. Solving this equation yields the parallel velocity, the total time derivative of which - unlike what is the case for the perpendicular derivative - has a finite convective part; the partial time derivative is taken to be zero.

Solving the equation of motion can be done as an initial value problem (in which case not only the equation for the velocity but also that for the position needs to be solved) or as a boundary value problem (in which case the time derivative is local and the solution is convected by the convection term). Two examples of the resolution of the equation of motion Eq. 9 as an initial value problem are depicted in Figs. 2 and 3, the former being an example for the ions and illustrating the force balance, the latter showing the velocities for electrons having velocities equal to 10% of the electron thermal velocity. In spite of being an equation for the 'slow' time scale and as earlier demonstrated when integrating the equation by hand, the solutions have a fast (cyclotron revolution) variation, imposed by the strong magnetic field superposed on a steady drift velocity. This rapid cyclotron motion can be isolated by splitting the equation of motion in a truly slow time scale varying part $\vec{V}_o$, and a rapidly oscillatory perpendicular part $\vec{v}$:
\[
\frac{\partial \vec{V}_o}{\partial t} + \frac{\partial \vec{V}}{\partial t} + \vec{V}_o \cdot \nabla \vec{V}_o + \vec{V} \cdot \nabla \vec{V}_o + \vec{V} \cdot \nabla \vec{V} + \vec{V} \cdot \nabla \vec{V} = \vec{a} + \vec{V}_o \times \vec{e} / \Omega + \vec{v} \times \vec{e} / \Omega
\]

Unlike what is commonly done when linearising equations there is no reason to assume the nonlinear terms are small: Quite on the contrary, the perpendicular gyro-oscillations commonly dominate the slower drifts by various orders of magnitude. Averaging this equation over a cyclotron period eliminates all terms linear in \(\vec{v}\) and yields

\[
\frac{\partial \vec{V}_o}{\partial t} + \vec{V}_o \cdot \nabla \vec{V}_o + < \vec{V} \cdot \nabla \vec{V} > = \vec{a} + \vec{V}_o \times \vec{e} / \Omega
\]

in which the period average of the rapidly varying quadratic term

\[
< \vec{v} \cdot \nabla \vec{v} > = \frac{1}{2} \left[ \left( \frac{1}{2} \frac{\partial v^2}{\partial x_{\perp,1}} - v^2 \frac{\partial \phi}{\partial x_{\perp,2}} \right) e_{\perp,1} + \left( \frac{1}{2} \frac{\partial v^2}{\partial x_{\perp,2}} + v^2 \frac{\partial \phi}{\partial x_{\perp,1}} \right) e_{\perp,2} \right]
\]

- where \(\vec{v} = e_{\perp,1} v_\perp \cos \phi + e_{\perp,2} v_\perp \sin \phi\) and \(\phi\) is the Larmor gyration angle - cannot generally be dropped: From the time dependent part of the earlier obtained result Eq. 10 we locally have \(\phi = \phi_o - \Omega t\) (we assumed \(\Omega\) is constant), from which it follows that \(< \vec{v} \cdot \nabla \vec{v} > = 0\) only if \(\phi_o\) and \(v_\perp\) are constant. For a sufficiently weak acceleration \(\vec{a}\) (which e.g. requires \(N_o^{-1} \nabla N_o\) to be small, which is not the case in the sheath) this still crudely holds true but in general the slow time scale equation requires knowledge of the fast time scale solution and the \(< \vec{v} \cdot \nabla \vec{v} >\) term influences the value and shape of \(\vec{V}_o\); corrections that are insignificant on the scale of \(\vec{v}\) represent major corrections to \(\vec{V}_o\) and cannot off-hand be neglected. Although the reason for the presence of the quadratic term is the same as what underlies quasilinear terms and the Ponderomotive force, the role of the term can be quite different. In the rather particular case where \(v_\perp\) and \(\phi_o\) are indeed constant, the relevant equation then reduces to Eq. 9, the equation we started from. The assumption that \(\vec{B}_o\) is strong so that \(\Omega\) is large makes that \(\partial / \partial t\) is typically of the order of the cyclotron frequency. Omitting it is a harsh limit and doing so to allow defining a meaningful steady state requires keeping the \(< \vec{v} \cdot \nabla \vec{v} >\) term since otherwise the 2-time-scale character of the physics (this paper explicitly aims at rapidly varying fields and strong magnetic fields) would be ignored. If on the other hand \(\vec{B}_o\) is weak, the Larmor gyration is slow and the Larmor radius is large, there is no immediate need for adopting a 2-time-scale approach advocated here and it makes sense to look for solutions of the steady state equation.
\[
\vec{V}_o.\nabla \vec{V}_o = \vec{a} + \vec{V}_o \times \vec{e}/\Omega. \tag{15}
\]

Adopting this steady state equation as such in the strong magnetic field limit artificially discards the net impact of the fast time scale effects on the slow time scale.

The parallel component of this equation is the above mentioned Eq. 14, in which the magnetic field dependence is absent. Integration of Eq. 15 to obtain the perpendicular dynamics yields excursions that are inconsistent with the results found when integrating the full time dependent Eq. 9. In view of the importance of the rapidly oscillating terms, the simplified ‘pure slow time scale’ steady state equation is not used to address the perpendicular dynamics. For that, the simpler but more general - and analytically derived - Eq. 11 i.e. \(\vec{a} + \vec{V}_o \times \vec{e}/\Omega = 0\) is adopted; henceforth \(\vec{V}_o = \vec{v}_{\text{drift},\perp} + \vec{v}/\Omega\). There is a caveat associated with that choice: the solutions found are only physically meaningful if a stationary state can be reached. The actual slow dynamics is then governed by

\[
\frac{d\vec{x}_{\perp}}{dt} = \vec{v}_{\text{drift},\perp} \tag{16}
\]

\[
\frac{d^2 x_{\parallel}}{dt^2} = \frac{dv_{\parallel}}{dt} = a_{\parallel} \tag{17}
\]

The corresponding velocities are equally depicted in the Figs. 2 and 3 for a toy problem with a prescribed potential and constitute the average motion around which the oscillatory motion occurs.

Unless when limiting oneself to a small number of cycles permitting using many time steps per period, brute-force integration of the full equation of motion proves to be numerically challenging, the slow dynamics terms being may orders of magnitude smaller than the fast dynamics and the loss in precision when evaluating the drift correction \(\vec{v}.\nabla \vec{v}\) accurately not permitting to follow the cyclotron oscillation motion on longer times scales. It would be desirable to re-address the issue of \(\vec{v}.\nabla \vec{v}\) and find a more suitable procedure which incorporates the effect of the net impact of all fast dynamics on the steady state more rigorously. That effort is reserved to later work but as an onset the full equation of motion

\[
m\frac{d\vec{v}}{dt} = q[\vec{E}_0 + \vec{E}_1 + \vec{v} \times (\vec{B}_0 + \vec{B}_1)]
\]

including the driven fields was integrated to determine the limits of the applicability of the time splitting procedure and of the adopted expression for the Ponderomotive force.
for a simple case: When focussing on the electrons and in the RF domain of frequencies
\(|\Omega_e| \gg \omega\), only the parallel electric field component needs to be retained as it can be seen
that \(E_{//}\) is the dominant component contributing to \(\Theta\). At first sight, it even comes as a
surprise that perpendicular gradients of the parallel electric field create a perpendicular drift.
Only including the electric field effects when solving the equation of motion of a particle in
a confining magnetic field in presence of a finite driven \(\vec{E}_1\) indeed yields results inconsistent
with the Ponderomotive force predictions. But aligned with the fact that in deriving \(\Theta\), \(\vec{B}_1\)
is eliminated in favour of \(\vec{E}_1\) using Faraday’s law, one realises that the perturbed magnetic
force term equally needs to be included. This adds contributions due to the presence of the
finite \(E_{//}\) to the magnetic force term. When doing so, the particle motion is recognised to
be the superposition of a fast motion and the perpendicular drift described by Eq. 11. As
can be expected from the details of the derivation, the method becomes questionable if the
truncated Taylor series expansion underlying the formulation of the Ponderomotive force
cannot sensefully be truncated after the linear term.

Another aspect of the discussion of importance of \(< \vec{v}.\nabla \vec{v}>\) is the fact that \(\Omega\) is constant
when referring to \(\vec{B}_o\) but is not when referring to the total magnetic field. Insofar as the RF
corrections can be neglected, \(\Omega\) is indeed essentially constant and from that it follows that
\(v_\perp\) and \(\phi_o\) are as well. But the just discussed example underlines that assessing the quality
of an approximation requires comparing with a suitable more general model. Moreover, for
the single particle equation of motion, the density does not enter the equation of motion
while for the fluid equation it does. In the latter case, the steepness of the density gradient
matters while in the former it does not.

B. Fast time scale wave equation

For a given density, the fast scale wave equation can be solved. It is implicitly assumed
that the static magnetic field is constant; on the Debye scale length this is a justifiable
assumption. \(\vec{B}_o\) can point in any direction, though. Two relevant Cartesian coordinate
frames \((x_\perp,1, x_\perp, 2, x_{//})\) and \((x, y, z)\) are connected by 2 consecutive rotations (see Fig.4).
First keeping the \(z\)-axis fixed rotating over the angle \(\alpha\) and then freezing the new \(y\)-axis and
rotating over \(\beta\) to lign up the final \(z''\) axis with \(\vec{B}_o\) yields
\[
\begin{pmatrix}
\vec{e}_{\perp,1} \\
\vec{e}_{\perp,2} \\
\vec{e}_{\parallel}
\end{pmatrix}
= \begin{pmatrix}
cos\beta \cos\alpha & \cos\beta \sin\alpha & -\sin\beta \\
-\sin\alpha & \cos\alpha & 0 \\
\sin\beta \cos\alpha & \sin\beta \sin\alpha & \cos\beta
\end{pmatrix}
\begin{pmatrix}
\vec{e}_x \\
\vec{e}_y \\
\vec{e}_z
\end{pmatrix}
= \mathbf{R} \cdot \begin{pmatrix}
\vec{e}_x \\
\vec{e}_y \\
\vec{e}_z
\end{pmatrix}.
\]

(18)

In this paper, the adopted dielectric response is modeled via the usual cold plasma dielectric tensor (see e.g.,\textsuperscript{37,38}) and hence the wave model is standard and requires no dedicated description. This choice - which ignores the presence of zero order flows when relating the driven velocity and the driving field - is dictated by the wish to keep the model as simple as possible. But since flows are an integral part of wave induced density modification, this choice needs to reviewed later so that the dielectric response in presence of flows is described. Up to a differing choice of the angles $\alpha$ and $\beta$, the specific implementation of the relevant wave equation was already commented on in\textsuperscript{18}.

Anticipating the variations in the $x$-direction (perpendicular to the metallic wall) dominate the variations in the 2 independent directions tangent to the wall, the $y$- and $z$-directions can be assumed ignorable for the zero order quantities and the variation of the rapidly varying electric field can be described by a double sum of decoupled Fourier modes ($k_y, k_z$). In terms of each ($k_y, k_z$), the wave equation can be written as

\[
\begin{pmatrix}
k_y^2 + k_z^2 & 0 & 0 \\
0 & k_z^2 & -k_y k_z \\
0 & -k_y k_z & k_y^2
\end{pmatrix}
+ \begin{pmatrix}
0 & ik_y & ik_z \\
ike_y & 0 & 0 \\
ike_z & 0 & 0
\end{pmatrix}
\frac{d}{dx}
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\frac{d^2}{dx^2}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
= k^2_0 \mathbf{R}^{-1} \mathbf{R}_{NR} \cdot \mathbf{R}
\]

(19)

in which $\mathbf{R}_{NR}$ is the usual non-rotated cold plasma dielectric tensor with respect to the $\vec{B}_o$-based directions ($\vec{e}_{\perp,1}, \vec{e}_{\perp,2}, \vec{e}_{\parallel}$). Since the cold plasma dielectric tensor is a matrix rather than a differential operator, the above equation can immediately be generalized to 2 or 3 dimensions by substituting ($k_y, k_z$) → $-i(d/dy, d/dz)$. To keep the computations fast, this will not be done in this paper.

As $E_x$ is a linear combination of the other 2 wave components and their first derivatives for each ($k_y, k_z$), it can be eliminated when solving the system. As $\mathbf{R}^{-1} = \mathbf{R}^T$, it is readily seen that the unit vectors along the coordinate lines transform as the corresponding vector components. Note that describing the variations along the magnetic field requires the
differential operator
\[
\vec{e}_{\parallel \parallel} \cdot \nabla = \frac{\partial}{\partial x_{\parallel \parallel}} = R_{31} \frac{d}{dx} + i R_{32} k_y + i R_{33} k_z = \cos \alpha \sin \beta \frac{d}{dx} + i \sin \alpha \sin \beta k_y + i \cos \beta k_z.
\]

Consequently, the parallel wave number is not a constant in general. As a result, the dispersion equation cannot be evaluated in the usual way by first solving for a prescribed \( k_{\parallel} \) to get the perpendicular wave numbers \( k^2_{\perp} \), from which subsequently the \( k_x \) values can be found. Only when \( \alpha = \pi/2 \) or \( \beta = 0 \), the parallel wave number is a constant. Assuming the waves are decoupled at the plasma interface, the cold plasma dispersion equation - found by substituting \( d/dx \rightarrow ik_x \) in the above wave equation - can be solved and the 4 possible kinds of waves (2 of the fast wave and 2 of the slow wave type) are identified. Rather than appearing as pairs \((k_x, -k_x)\) corresponding to a common \( k^2_x \) for a prescribed \( k_{\parallel} \), the 4 roots of the dispersion equation are typically distinct.

Four boundary conditions are required to uniquely define the solutions of this fourth order differential equation. The boundary conditions at the metallic wall are \( E_y = E_z = 0 \). The other 2 boundary conditions are imposed at the interface \( x_p \) of the (sheath) region of interest with the deeper plasma. Assuming the various modes supported by the plasma are decoupled at the plasma interface, solving the polarization equation for each of the wave types yields the eigenvectors. The independent variables and the eigenvectors \( \vec{s}_i \) are related by the matrix \( \overline{T} \):

\[
\begin{pmatrix}
E_y \\
dE_y/dx \\
E_z \\
dE_z/dx
\end{pmatrix} = \overline{T} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix}
a_1 & a_2 & \cdots \\
a_1 ik_{x,1} & a_2 ik_{x,2} & \cdots \\
a_1 E_z/E_y|_1 & a_2 E_z/E_y|_2 & \cdots \\
a_1 ik_{x,1} E_z/E_y|_1 & a_2 ik_{x,2} E_z/E_y|_2 & \cdots
\end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \tag{20}
\]

The coefficients \( a_i \) are guaranteeing that the electric field components of each of the reference eigenvectors corresponding to the roots \( k_{x,i} \) have unity amplitude. At any location the full electric field is a combination of fast (FW) and slow waves (SW). Referring to the various eigenvectors, a natural choice for the boundary conditions is to impose that only 1
kind of wave carries energy into the sheath. This then e.g. requires $s_{FW,\leftarrow} \neq 0$ and $s_{SW,\rightarrow} = 0$ or $s_{FW,\rightarrow} = 0$ and $s_{SW,\rightarrow} \neq 0$. Solving the wave equation can equally be done imposing only 1 type of outgoing wave carries energy away from the sheath. Clearly these solutions are not independent of the 2 already found. A set of 4 complex connection coefficients that relate the eigenvector amplitude of the incoming and outgoing waves for both kinds of excitation is found. They form a matrix that constitutes a representation equivalent to the surface impedance matrix often used in antenna modelling.

The adopted set of boundary conditions is useful for demonstrating the interplay between the different modes supported by the plasma in the sheath. For later, more practical applications more convenient boundary conditions (e.g. matching 2 of the relevant electric field components, which is easier to implement) are likely to be preferred.

From the expression for the Ponderomotive acceleration it is readily seen that the various waves supported by the plasma impact differently on the density. Figures 5 and 6 show the roots of the dispersion equation and the corresponding exponent of the density expression for the earlier considered parameters and for the 2 limiting cases $\alpha = \beta = 0.01\,\text{rad}$ (representative for the magnetic field nearly parallel to the wall) and $\alpha = 0.01\,\text{rad}$ and $\beta = 1.57\,\text{rad}$ (field nearly perpendicular to the wall) when scanning the density. It is readily seen that the slow wave has a much more pronounced impact than the fast wave has and that this is an immediate consequence of the difference in the magnitude of their respective wave number. On the scale of the slow wave, the fast wave is hardly noticed on the top plots showing the 4 dispersion equation roots. The fact that the dispersion equation has odd order terms making that the roots do not appear as pairs $(k_x, -k_x)$ is clearly seen. The lower hybrid resonance is recognised as the region where the slow wave becomes propagative in Fig. 5 for densities a bit lower than $10^{17}/\text{m}^3$. It was already discussed in previous work that the presence of the lower hybrid resonance has repercussions both on the physics of the wave dynamics in the sheath as on the numerical challenge arising when incorporating it in the model. When the slow wave physics is non-essential, analytical models of the Budden type can be invoked to avoid needing to resolve the slow wave structure, replacing it by a local power sink. But in the physics of the sheath - and more general in the low density region in which the wave launcher is embedded - the slow wave physics is an essential ingredient. As expected, the impact of the slow wave on the density is much more pronounced when the wave is evanescent. Note that the depletion is in the direction of the wave propagation.
The shown curves are somewhat misleading: The amplitude of the wave field is imposed here, while in practice waves can only have an impact if they are actually excited. Further more, the global effect on the density is the result of the sum of all waves and not that of a single mode. This is particularly important close to the metal wall, from which waves are nearly totally reflected when incident on it. In practice, the density depletion is more modest than suggested by the curves for the individual wave modes because of the fact that generally several modes coexist to ensure the boundary condition is satisfied by the total $\vec{E}$-field. When discussing the solution of the full problem later in the text, this effect will be accounted for. The here presented curves mainly give an idea of the potential of waves to modify the density.

C. Continuity equation

Making an identical splitting of fast and slow time scales as was done for the equation of motion, the relevant continuity equations are obtained. The slow time scale continuity equation reads

$$\frac{\partial N_o}{\partial t} + \nabla . (N_o \vec{v}_o) + \frac{1}{2} \Re \left[ \nabla . (N_1^* \vec{v}_1) \right] = 0. \quad (21)$$

in which the perturbed velocity and density appear in the last term; this quasilinear term is the net contribution of the fast time scale wave dynamics to the slow time scale equation. The perturbed velocity was already specified earlier. Conform with the earlier made assumptions, the slow time scale continuity equation will be solved in the limit $\partial / \partial t \to 0$ so one gets

$$N_o \frac{\partial v_{//}}{\partial x_{//}} + v_{//} \frac{\partial N_o}{\partial x_{//}} + \vec{v}_{\perp,o} \cdot \nabla N_o + \frac{1}{2} \Re \left[ \frac{d}{dx} (N_1^* v_{1,x}) \right] = 0 \quad (22)$$

in which $\nabla . \vec{v}_{\perp,o} = 0$ was used and where the last term in the above only contains the $x$-derivative term since the perturbed quantities vary as $\exp[i(k_y y + k_z z - \omega t)]$ in the directions perpendicular to the wall and hence quadratic quantities of the form $\Re[A^* B]/2$ are independent of $y$ and $z$ when a single set $(k_y, k_z)$ of modes is looked at. Note that in absence of RF waves and if the confining magnetic field is strong, $N_o v_{//}$ is roughly a constant. In that limiting case, the parallel current across the sheath is constant. More generally it is not:
due to the gradients, current can partly flow sideways and does not need to flow straight into the wall.

Using the equation of motion to eliminate the parallel velocity gradient, the above can be reformulated. The result is

\[ v_\parallel^2 - v_\parallel^2 \frac{\partial \ln N_o}{\partial x_{\parallel}} + v_\parallel v_\perp \nabla_\perp \ln N_o - \frac{q}{m} \frac{\partial \Phi}{\partial x_{\parallel}} - \frac{\partial \Theta}{\partial x_{\parallel}} + \frac{v_{\parallel}}{2N_o} \text{Re} \left[ \frac{d}{dx} N_1^* v_{1,x} \right] = 0. \] (23)

The coefficient in front of the leading term states that the density grows or decays depending on whether the flow is faster or slower than the thermal flow. In absence of a rapidly varying field and for a strong static magnetic field, this equation reduces to the one already adopted in\textsuperscript{18} where the wave equation was solved without accounting for the wave induced corrections on $N_o$. Whereas the parallel flow equation suggests that the ion density can grow very large when the electron density is dropping, the combined equation learns that ion flow velocities satisfying the Bohm criterium equally decay in the sheath. As the electron flow velocity is much smaller than the electron thermal velocity in the sheath, the Boltzmann expression is recaptured in that particular limit.

An expression for the perturbed velocity was evaluated in the previous section. The perturbed density obeys the fast time scale continuity equation,

\[ -i\omega N_1 + \nabla \cdot [N_o \vec{v}_1 + N_1 \vec{v}_0] = 0. \] (24)

It may be argued - rightfully - that the slow time scale continuity equation should in general allow for source/loss terms so that the density variation is not just due to convection. Especially in the plasma edge processes such as ionisation, recombination or recycling are likely to be effects that have an impact on the density profile reached. In the present paper - focussing on the dynamics in the limited volume of the sheath - these effects are implicitly assumed to be accounted for by the prescribed plasma composition. When modelling the wave-particle interaction more macroscopically (e.g. looking at the density near the launcher and not just in the sheath) such effects can no longer be discarded. A common example for tokamak applications is the cross-field diffusion that leaks particles across the last closed flux surface so that the source feeding particles to a magnetic field line is of the form $\partial / \partial x_\perp [D_\perp \partial / \partial x_\perp [N_o]]$ where $x_\perp$ is the magnetic surface labeling parameter, to be identified
with one of the 2 variables $x_{\perp,1}$ or $x_{\perp,2}$ introduced before (see e.g.\textsuperscript{24}). A qualitative estimate of the solution of the simplest possible steady state continuity equation with source

$$
\frac{\partial}{\partial x_{//}}[v_{//}N_o] = \frac{\partial}{\partial x_{\perp}} D_{\perp} \frac{\partial}{\partial x_{\perp}} N_o
$$

then shows that the density is essentially proportional to $exp[\alpha_{\perp} x_{\perp}]$ where $\alpha_{\perp} = (v_{//}/[D_{\perp}L_{//}])^{1/2}$ in the perpendicular direction, while in the parallel direction it varies crudely as $exp[\alpha_{//} x_{//}]$ with $\alpha_{//} = D_{//}/[L_{//}^2 v_{//}]$. The exponential decay length across the magnetic field lines behind the last close flux surface is typically of order of a few centimeter: for a deuterium plasma, assuming an edge temperature of $T = 10eV$, taking the parallel scale length $L_{//}$ to be $10m$ and $D_{\perp} = 1m^2/s$, one gets $1/\alpha_{\perp} \approx 0.02m$. On the Debye length scale of the sheath dynamics this is very very long. The density gain or loss through ionisation or recombination is described via the reaction rate of the process modeled and involves the densities of the species taking part in the interaction i.e. $\partial N_i/\partial t|_{k,ij} = \pm R_{k,ij} N_i N_j$ where the '+' is relevant for the 'receiving' specie and the '-' for the 'donating' specie. Incorporating a large set of relevant processes and their respective reaction rates, Wauters successfully adopted this approach to develop a model for the birth of plasma under the influence of RF waves to get a grip on the dynamics of ion cyclotron wall conditioning and RF assisted plasma startup\textsuperscript{40,41}. The so far described sources scale with the density. Aside from this kind of sources, the density may change due to processes that are independent of the local density. When aiming at making realistic macroscopic predictions on wave induced density depletion all such effects should be included in the model.

D. Slow time scale wave equation: Poisson’s equation

Since the slow time scale magnetic field is constant in time, the electric field can be derived from a potential. The law of Gauss then reduces to the Poisson equation:

$$
\Delta \Phi = \frac{e}{\epsilon_o} [N_{e,o} - \sum_i Z_i N_{i,o}]
$$

(25)
IV. ADOPTED EQUATIONS AND NUMERICAL PROCEDURE

Due to the presence of the strong, static and constant magnetic field and the fact that the magnetic field cannot accelerate particles along its field lines, the slow time scale equation of motion can be solved explicitly for the perpendicular aspects of the motion forced by $\vec{B}_o$ while the parallel motion is a differential equation with $x_{//}$ as the independent variable. On the other hand, the presence of a metallic wall yields sheath dynamics essentially in the $x$-direction perpendicular to the wall. It will be assumed in this paper that the slow time scale quantities’ characteristic gradients lengths along the wall are much smaller than those perpendicular to it so that the former can be assumed to be locally constant ($\partial/\partial y \approx \lambda_y$ and $\partial/\partial z \approx \lambda_z$) and sufficiently small to justify the use of single (uncoupled) sets ($k_y, k_z$) in the fast time scale equation i.e. assuming these directions are quasi-ignorable. Strictly, the earlier described equations allow application of the proposed method to 2D or 3D application and thus allow to have the full ($k_y, k_z$) spectrum for the fast time scale and proper derivatives for the slow time scale variables, doing away the need to impose prescribed variations along the metallic wall.

The differential operators needed in the various slow time scale equations now reduce to

$$
\begin{pmatrix}
\frac{\partial}{\partial x_{//,1}} \\
\frac{\partial}{\partial x_{//,2}} \\
\frac{\partial}{\partial x_{//}}
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{R}_{11} \\
\mathcal{R}_{21} \\
\mathcal{R}_{31}
\end{pmatrix}
\frac{d}{dx} +
\begin{pmatrix}
\mathcal{R}_{12} \\
\mathcal{R}_{22} \\
\mathcal{R}_{32}
\end{pmatrix}
\lambda_y +
\begin{pmatrix}
\mathcal{R}_{13} \\
\mathcal{R}_{23} \\
\mathcal{R}_{33}
\end{pmatrix}
\lambda_z.
$$

(26)

and hence the relevant equations are:

- The slow time scale continuity yielding $v_{//}$:

$$
A_{v_{//}}(x)\frac{dv_{//}}{dx} + B_{v_{//}}(x)v_{//}} + C_{v_{//}}(x) = 0
$$

(27)

where

$$
A_{v_{//}}(x) = \mathcal{R}_{31}N_o
$$

$$
B_{v_{//}}(x) = \mathcal{R}_{31} \frac{dN_o}{dx} + \frac{3}{2}(\mathcal{R}_{32}\lambda_y + \mathcal{R}_{33}\lambda_z)N_o
$$

$$
C_{v_{//}}(x) = \left[v_{\text{drift,} \perp 1}\mathcal{R}_{11} + v_{\text{drift,} \perp 2}\mathcal{R}_{21}\right] \frac{dN_o}{dx}
+ \left[v_{\text{drift,} \perp 1}(\mathcal{R}_{12}\lambda_y + \mathcal{R}_{13}\lambda_z) + v_{\text{drift,} \perp 2}(\mathcal{R}_{22}\lambda_y + \mathcal{R}_{23}\lambda_z)\right] N_o
$$
A 'density source' term $-D_\perp/\lambda^2_{SOL} N_o$ where $D_\perp$ is the diffusion across magnetic field lines and $\lambda_\perp$ is a typical decay length of the density in the scrape-off layer can be added to $C_{v_{//}}$ when modelling macroscopic decay. As boundary condition it seems natural to assume that the parallel flow is known at the entrance of the region of interest. In the spirit of sheath dynamics, $v_{//}$ is taken to be of order of the thermal velocity for the ions while the electron flow is chosen to guarantee that no steady state current enters the sheath i.e. $v_{//,e} N_e = -\sum_i Z_i N_i v_{//,i}$.

- The slow time scale parallel equation of motion yielding the (logarithm of the) density $N_o$:

$$A_{\ln N_o}(x) \frac{d\ln N_o}{dx} + B_{\ln N_o}(x) \ln N_o + C_{\ln N_o}(x) = 0$$

(28)

where

$$A_{\ln N_o}(x) = v_t^2 R_{31}$$

$$B_{\ln N_o}(x) = v_t^2( R_{32} \lambda_y + R_{33} \lambda_z)$$

$$C_{\ln N_o}(x) = R_{31} \frac{d}{dx} [\frac{q}{m} \Phi + \Theta] + [R_{32} \lambda_y + R_{33} \lambda_z] [\frac{q}{m} \Phi + \Theta]$$

$$+ (v_{\text{drift,} \perp 1} R_{11} + v_{\text{drift,} \perp 2} R_{21} + v_{//} R_{31}) \frac{\partial v_{//}}{\partial x} + \frac{1}{2} (R_{32} \lambda_y + R_{33} \lambda_z) v_{//}}^2$$

$$+ \frac{1}{2} [v_{\text{drift,} \perp 1}(R_{12} \lambda_y + R_{13} \lambda_z) + v_{\text{drift,} \perp 2}(R_{22} \lambda_y + R_{23} \lambda_z)] v_{//}}$$

in which $\lambda_y$ and $\lambda_z$ are assumed to be the common local derivatives of $N_o$, $v_t^2$, $\Phi$ and $\Theta$ in the $y$ and $z$ direction, respectively. Integrating the logarithm of the density rather than the density itself proved to be numerically more stable. It is assumed that charge neutrality is guaranteed at the interface with the main plasma. As boundary condition, the electron and ion densities are imposed at that location. Consistent with the alternative Eq. 23, the above can be substituted for

$$A_{\ln N_o}(x) = [v_{//}}^2 - v_t^2] R_{31} + v_{//} [v_{\text{drift,} \perp 1} R_{11} + v_{\text{drift,} \perp 2} R_{21}]$$

$$B_{\ln N_o}(x) = [v_{//}}^2 - v_t^2] [R_{32} \lambda_y + R_{33} \lambda_z] + v_{//} [v_{\text{drift,} \perp 1} (R_{12} \lambda_y + R_{13} \lambda_z) + v_{\text{drift,} \perp 2} (R_{22} \lambda_y + R_{23} \lambda_z)]$$
\[ C_{lnN_o}(x) = -v_{drift,\perp 1} \left[ R_{11} \frac{\partial v_{\perp 1}}{\partial x} + \frac{v_{\perp 1}}{2} \left( R_{12} \lambda_y + R_{13} \lambda_z \right) \right] - v_{drift,\perp 2} \left[ R_{21} \frac{\partial v_{\perp 2}}{\partial x} + \frac{v_{\perp 2}}{2} \left( R_{22} \lambda_y + R_{23} \lambda_z \right) \right] \]
\[ - \frac{q}{m} \left[ R_{31} \frac{\partial \Phi}{\partial x} + \left( R_{32} \lambda_y + R_{33} \lambda_z \right) \Phi \right] - R_{31} \frac{\partial \Theta}{\partial x} + \frac{v_{\perp 1}}{2N_o} Re \left[ \frac{d}{dx} (N_1^* v_{1,x}) \right] \]

Note that the last term in the above points to a weakness in the model: when \( N_o \to 0 \), \( C_{lnN_o} \to \infty \). Rather than being physically meaningful, this divergence is a testimony of the fact that the linearisation - which assumes \( N_1 << N_o \) - breaks down and needs to be replaced by a suitable alternative.

- The slow time scale wave equation yielding the potential \( \Phi \):

\[ \frac{d^2 \Phi}{dx^2} + (\lambda_y^2 + \lambda_z^2) \Phi = \frac{e}{\epsilon_o} [N_e - \sum_i Z_i N_i] \]  (29)

For given \( N_o \), imposing \( \Phi = 0 \) at \( x_p \) (the sheath ends where charge neutrality is re-captured) and a finite value \( \Phi = \Phi_{wall} \) at \( x_{wall} \) yields a unique solution. This set of boundary conditions is useful in case some external mechanism (e.g. grounding) is imposing the voltage on the vessel wall. Quite in general, that voltage is a free parameter, however. Other choices were considered as well. To ensure the potential reaches a constant value at the interface with the plasma, imposing a vanishing derivative of \( \Phi \) was for example also a handy boundary condition to ensure charge neutrality is gradually removed towards the interface with the rest of the plasma.

For reaching a steady state it is necessary to have zero current flowing towards the wall since otherwise - in absence of external charges flowing towards it - the wall potential cannot reach a stationary value. Poisson’s equation does not allow imposing a current as boundary condition. It does not even distinguish between electron and ion charges but has the summed charged density on its right hand side. One can only impose a potential, the derivative of a potential or a linear combination of these 2 at some reference place(s) as conditions to have a unique desired solution. The (charge) density is an input and the velocity does not even enter the picture so in itself the slow time scale current cannot be imposed i.e. the condition for absence of flow towards the wall needs to be done via a procedure external to Poisson’s equation. Setting up an iterative scheme allows finding a solution for which not only the current at the plasma side (imposed to be zero by choosing the edge parallel velocities) but also that at the wall vanishes.
From a series of possible values for the wall potential for each of which the nonlinear system of equations is solved, the particular steady state for which the zero current condition is also satisfied at the wall is one of the possible solutions, and the physically desired one. Admittedly, this issue requires further attention: the presently adopted trial-and-error adjustment is not very satisfactory.

- The fast time scale wave equation yields the perturbed electric field $\vec{E}$; details can be found in the dedicated section. Once $\vec{E}$ is known, the components of the perturbed velocity can be computed, and $\Theta$ can be found. The perturbed density follows from the fast time scale continuity equation,

$$A_{N_1}(x) \frac{dN_1}{dx} + B_{N_1}(x)N_1 + C_{N_1}(x) = 0$$  \hspace{1cm} (30)

with

$$A_{N_1}(x) = v_{\text{drift},\perp 1} R_{11} + v_{\text{drift},\perp 2} R_{21} + v_{//} R_{31}$$

$$B_{N_1}(x) = R_{31} \frac{dv_{//}}{dx} + i \left[ -\omega + v_{\perp,1} [k_y R_{12} + k_z R_{13}] + v_{\perp,2} [k_y R_{22} + k_z R_{23}] + \frac{v_{//}}{2} [k_y R_{32} + k_z R_{33}] \right]$$

$$C_{N_1}(x) = N_o \left[ \frac{dv_{1,x}}{dx} + i k_y v_{1,y} + i k_z v_{1,z} \right] + v_{1,x} \frac{dN_o}{dx} + (v_{1,y} \lambda_y + v_{1,z} \lambda_z) N_o$$

A suitable option for the boundary condition of this first order equation is to assume that the density $N_1$ is known as some position. It will be assumed that the density at the interface with the main plasma is known. Since the wave field modifies the density and one expects the deviation to be larger for larger fields, one can - however - not simply force $N_1$ to have a prescribed density. As will be explained next, setting up an iterative scheme not only allows to solve this issue but equally provides an elegant solution for tackling the set of equations at hand.

Various of the equations are linear equations for individual quantities but are nonlinear equations in terms of the various unknown quantities. Rather than solving all (linear and nonlinear) equations simultaneously, the set of equations can be solved iteratively addressing the dependence of the variables one by one and hence solving a series of linear equations at each iteration. First the solution is found in absence of RF fields and from an approximate density profile $N_o$ for the various species. Once the self-consistent solution is found in absence of RF heating, the heating is switched on and adiabatically increased. Once the desired electric field level is reached, iterations are done until convergence is reached.
Aside from the fast and slow time scale wave equations, all relevant equations are first order differential equations of the form

$$A \frac{dF}{dx} + BF + C = 0.$$  \hspace{1cm} (31)

The solution of this first order differential equation is

$$F(x) = [F(x_{\text{ref}}) - \int_{x_{\text{ref}}}^{x} dx \frac{C}{A} \exp[\int_{x_{\text{ref}}}^{x} \frac{B}{A} dx]] \exp[- \int_{x_{\text{ref}}}^{x} \frac{B}{A} dx].$$  \hspace{1cm} (32)

Once $F$ is known, the derivative of the solution is directly given from the original equation.

V. A FEW PRELIMINARY EXAMPLES

Although the discussion so far should have made clear that the physics of wave-plasma interaction cannot be captured satisfactorily in a 1D model, a few examples are discussed in this section. Unless specified otherwise for specific details, the following parameters were considered: 5% hydrogen in a deuterium majority plasma, electron and ion temperature $T_e = T_i = 15eV$, magnetic field strength $B_o = 2.5T$ at the antenna and a driver frequency of $f = 51MHz$. Waves hitting the metallic wall of the vessel are considered but the magnetic field is not required to be parallel to the wall. For the presented example the magnetic field direction is determined by $\alpha = \beta = 0.5rad$, on the fast time scale the waves enter the region of interest obliquely to the wall picking $k_y = 5/m$ and $k_z = 6/m$ while the slow time scale variations of the parallel energy, the density and the electrostatic potential along the wall are prescribed via $\lambda_y = \lambda_z = 0.1/m$.

The first examples illustrate the dynamics in the sheath formed close to a metallic wall. A density of $N_e = 10^{17}/m^3$ was chosen and the considered integration interval is 15 Debye lengths wide. Figure 7 depicts the density profiles of the electrons, majority D ions and minority H ions for various values of the parallel velocity imposed at the right edge of the integration interval where the sheath connects to the main plasma. The ion velocities are prescribed to be the thermal velocity multiplied by a factor; the electron parallel flow velocity was chosen to ensure that there is no slow time scale current density at the sheath entrance ($J \approx -eN_e v_{/} + \sum_i Z_i N_i v_{/;i} = 0$). Whereas the electron density more or less returns to a common value at the wall, the wall ion density grows with the initial speed. This behaviour can easily explained from the factor $v_{/}^2 - v_i^2$ in front of the density derivative.
in the equation prescribing the density change. In the present paper the RF induced density changes are described as a 1D problem, whereas the problem at hand is actually 3D in nature. That the density profile equally depends on the variations in the $y$- and $z$-directions is illustrated in Fig. 8. The stronger the gradients parallel to the metallic wall are, the higher the densities tend to stay. As before the densities and parallel velocities are assumed known at the interface of the sheath with the rest of the plasma.

Figure 9 illustrates how the electric field modifies the density profile. Larger electric field amplitudes result in a more significant density depletion near the wall. For the adopted parameters ($v_{//}/v_{th,i} = 1.5$), both the electron and the ion density equally degrade towards the wall.

As a final example, the macroscopic radial density depletion in the antenna region as predicted by the here presented model is given in Fig. 10. As mentioned when discussing the model, this illustration is very preliminary (i) since the physics of RF induced flows is intrinsically at least 2-dimensional (see e.g. the flow along as well as perpendicular to the antenna due to the radial and poloidal components of the gradient of the field magnitude and thus of the drifts perpendicular to $\vec{B}_o$ and to the gradient caused by the Ponderomotive force along the antenna and at its tips as illustrated in\textsuperscript{17}) and (ii) since no assessment was made in this paper on which source terms (ionisation, recombination, diffusion, ...) are the most essential contributors to trustfully capture the dynamics close to the wave launcher and thus beyond the last closed flux surface. Nevertheless, it seems useful to illustrate the isolated impact of the presence of RF waves on the density close to the antenna, to get a feeling of the magnitude of the effect. For this example charge neutrality was assumed to hold: the microscopic sheath close to metallic walls was neglected and Poisson’s equation was not solved. Also the slow time scale gradients parallel to the wall were assumed to be negligibly small. The antenna is located in the centre of the considered integration interval and is modeled as a simple current sheet. The metallic back wall of the box is 10cm away from the antenna to the left, and the main plasma (antenna box mouth) is 10cm to the right of it. At the latter, the density is prescribed. A value of $N_e = 5 \times 10^{17}/m^3$ was taken. The density modifications for 4 different values of the angle $\alpha$ were depicted. A common electric field strength of 150kV/m was considered (recall that voltages of a few tens of kV are routinely imposed on RF antennas and that the - usually grounded - side and back walls of the antenna box are often only several cm away from the straps). Aside from lacking
physics processes, there is a geometrical reason why the present 1D model is inadequate to capture the full dynamics of the physics: being modeled as a current sheet, the plasma can freely flow through the antenna in the present description whereas in reality it needs to flow around it.

For the highest values of $\alpha$ the magnetic field is nearly parallel to the metallic wall; the limiting case $\alpha = \pi/2$ cannot be modeled since then there is no projection of $x_{//}$ onto $x$ and hence the adopted equation for the parallel flow cannot be used to predict parallel flow modifications perpendicular to the wall; in that particular case knowing the density at the edge does not allow predicting its value elsewhere. The density depletion observed in the figure for $\vec{B}_o$ nearly parallel to the wall is partly the consequence of the geometrical fact that the distance traveled along the magnetic field line is much larger than the distance traveled perpendicular to the wall. Once more this points towards the need of extending the model to more than 1 dimension: the actual distortion of a flow can be rather different from the flow predicted by a model that assumes that the variations along the wall are insignificant...

VI. CONCLUSIONS AND DISCUSSION

In the present paper a model is presented to study the interaction of RF waves and a plasma. The focus has been on the microscopic description in the sheath region, although all equations are given for 2D or 3D application at the start. In view of the vastly different time scales involved, a 2-time scale model has been set up. On both time scales it consists of a wave equation, an equation of motion, and the continuity equation. The slow time scale dynamics is influenced by the fast time scale physics through quasilinear modifications. The fast time scale equation of motion is solved analytically as a driven problem, and the corresponding wave fields and fast time scale density modification are computed starting from this same assumption. The slow time scale equation of motion is equally solved analytically for what concerns the perpendicular dynamics - implicitly assuming the static magnetic field is strong and that its gradients can be neglected when seeking a solution for the wave-plasma interaction in a sufficiently small region - while the parallel equation of motion needs to be solved numerically, except for the case when drifts can be neglected. Neglecting drifts altogether and just focussing on the parallel dynamics allows to integrate the equation of motion
by hand and yields a generalisation of the Boltzmann density expression commonly adopted for the electrons. In that expression the parallel energy, the electrostatic potential and the Ponderomotive potential play a similar role. Just like the dependence of the electrostatic potential on the sign of the charge makes that ions and electrons respond differently, the difference in mass of these species has a direct implication on the magnitude of the Ponderomotive potential for the different species. Moreover, the required wave polarisation has repercussions on which species are affected: For frequencies in the ion cyclotron frequency range, the electron dynamics is dominantly controlled by the parallel electric field while the ions are mainly affected by the perpendicular components. To the exception of the case where an ion cyclotron resonance is crossed in the region of interest (for which case strictly a kinetic model - see e.g.\textsuperscript{42,43} - is required as the cold plasma expressions artificially predict an infinite response), the Ponderomotive effect is not as important for ions as it is for electrons for the influence arising from the motion along the magnetic field lines. The adopted fast time scale wave equation is the standard one; it relies on the cold plasma dielectric tensor but takes the actual densities into account for the electrons as well as the ions. For the slow time scale wave equation it is assumed that a steady state has been reached so that Faraday’s law states that the corresponding electric field is the gradient of a potential. As a consequence, the full electromagnetic equation does not need to be solved on that time scale but can be replaced by Poisson’s equation consistent with a charge separation.

The main result of the paper is that the earlier found expression for the Ponderomotive potential in presence of a strong magnetic field (see e.g.\textsuperscript{17,36}) plays a role similar to the electrostatic potential for wave induced density modification. In reference\textsuperscript{17,36} the accent was on the perpendicular drifts brought about by the Ponderomotive force and allowing to explain that the effect of this force is that variations of the electric field amplitude perpendicular to the antenna straps cause poloidal drifts along it and that the strong gradients at the antenna strap tips cause radial flows towards and away from the antenna. In that paper (which described effects that are beyond a 1D description), the back-reaction of the electric field through the Ponderomotive potential was not incorporated in the density modification. This back-reaction was the main subject of the present paper, be it that the model was kept 1-dimensional here so that variations of fast as well as slow time scale variables along metal walls are prescribed rather than computed selfconsistently. Ions and electrons are treated on the same footing but respond differently to the presence of the potentials, electrons being
chased out of the sheath region and ions being attracted to it while also sensing a comparable contribution from the thermal motion. At the plasma side, it is imposed that no net current flows into the sheath. At the wall the annihilation of the current - needed to be able to set up a steady state which stops the wall from being charged further to more negative potentials is the result of the response of the species to the modifications of the potentials close to the wall.

The resulting set of equations is a mix of linear and non-linear equations. Rather than solving the actual nonlinear problem, an iterative scheme was set up to solve the non-linear problem as a series of linear ones. The various gradients in the equations of motion, and the continuity equation are gradually switched on so that the parallel velocity, density, potential and field can be approximated at each step by the value found in the previous iteration. Adding supplementary iterations after the various derivatives are at full strength allows reaching a steady state solution. In view of the non-linearities, the equations solved remaining numerically touchy, however, which requires many iterations (of the order of 1000) before reaching a steady state, and equally requiring a high number of grid points. For the present examples up to 60000 grid points are taken for the first order equations; the wave equations typically require only a few hundred grid points. The difficulty of solving the nonlinear set of equations at hand has already been noticed: Van Compernolle\textsuperscript{44} experimented with implementing the sheath boundary condition due to Myra and D’Ippolito - as mentioned at the outset an extremely useful approach enabling to avoid CPU intensive computations when doing macroscopic computations in more than 1 dimension but subject to a number of simplifying assumptions - and found satisfactory results at modest power levels but diverging results when the sheath width grew unrealistically large at higher, more realistic power levels. This divergence is likely linked to the fact that the linearisation assumption (and in particular $N_1 << N_o$) breaks down and points to the need for developing a suitable model for when that happens.

Further upgrading the model is a must. The here presented set of equations has been applied in 1D to focus on the dynamics close to a metallic wall. Retaining rather than neglecting the omitted derivatives and adding proper 'source' terms would allow to assess the dynamics in 2D or 3D and can be done in a macroscopic region rather than just in the small sheath region close to a metallic wall. This would offer a method for describing the wave induced density depletion in the antenna box, as was demonstrated by a first, rather
crude example. Such an approach was already tested in\textsuperscript{17} without iterating on the obtained density and fast time scale wave field. Since zero order flows are formed, the cold plasma dielectric tensor (which assumes $\vec{v}_o = 0$) can strictly no longer be used so, ultimately, also the fast time scale wave equation needs to be upgraded to the set of equations mentioned in that paper. For the specific focus on the dynamics in the sheath, including an energy equation - enabling to track the temperature changes assumed absent in the present paper - seems a plausible extension that needs to be made. More work to upgrade the physics model and to include 2 and 3D effects is reserved for future work. To name just 1 example: The steep gradients that are formed make the adopted iteration procedure subject to trial and error. Where such a brute force procedure is already not really satisfactory for the here present result, it becomes prohibitive when trying to tackle the equations in more than 1 dimension.

As a finishing note, one needs to remind that the here presented model has the strengths but also the weaknesses that come along with a fluid model. Kinetic modelling remains necessary to obtain more detailed insight in certain aspects that involve deviations away from Maxwellian distributions. The main advantage of fluid modelling compared to kinetic modelling is that it requires less computer time and computer memory. Another obvious limitation is that the derivation of the Ponderomotive force rests on the assumption that the Taylor series expansion of the field can be truncated after the linear term. In the sheath this may be questioned, a weakness not only suffered by the fluid type approach adopted here. In the search for a more satisfactory model, various of these basic assumptions need to be reassessed and the impact of their violation be quantified. Basic problem for such an assessment is that - by definition of the need to the type of work reported on in this paper - there is not yet a bullet-proof model that solves the complete equations fully rigorously and hence assessing the magnitude of errors made can only be done by comparing the predictions of different methods that all have their weaknesses and strengths ...

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FIG. 1. Electron and (majority) ion density modification along magnetic field lines as a result of a $v_{||}$ or potential change.
\(\alpha = \beta = 5 \times 10^{-1}, B_0 = 3.4 \text{T}, ^3\text{He}, T = 5 \text{eV}, v_{x,0} = v_{y,0} = v_{z,0} = v/3\)

FIG. 2. Solution of the slow time scale equation of motion for \(^3\text{He}\) ions; for this toy problem test \(\vec{a}\) is assumed to be the gradient of a potential \(\psi = 10^6(2x^4 + 5y^4 + 10z^4)(\text{m/s})^2\); \(\alpha = \beta = 0.5\text{rad}\).

\(\vec{v}_{\perp,1}, \vec{v}_{\perp,2}, \vec{v}_//, v_{x,s}, v_{y,s}, v_{z,s}\)

FIG. 3. Electron total as well as slow time scale (drift) velocities obtained solving Eq.8 or Eq.16; the magnetic field magnitude and direction is as in the previous example.
FIG. 4. Sheath \((x, y, z)\) and magnetic field \((x'', y'', z'')\) based coordinate frames.
FIG. 5. Dispersion roots, Ponderomotive potential and density decay rate as a function of the log$_{10}$ of the density for $\alpha = \beta = 0.01\text{rad}$.
FIG. 6. Dispersion roots, Ponderomotive potential and density decay rate as a function of the log$_{10}$ of the density for $\alpha = 0.01\,\text{rad}$ and $\beta = 1.57\,\text{rad}$. 
FIG. 7. Density profile inside the sheath in absence of RF fields for various values of the parallel velocity at the interface of the sheath with the main plasma (where charge neutrality is satisfied).
FIG. 8. Density profile inside the sheath in absence of RF fields for various values of the inhomogeneity strengths $\lambda = \lambda_y = \lambda_z$ in the directions parallel to the metallic wall.
FIG. 9. Density profile inside the sheath for various RF electric field strengths imposed at the interface of the sheath with the main plasma (where charge neutrality is satisfied) in the directions parallel to the metallic wall.
FIG. 10. Macroscopic RF induced density depletion for various angles of $\alpha$. 
$N_0(\nu_{//}, \Phi)/N_0(0,0):$ electrons
$N_0(v_{\parallel}, \Phi)/N_0(0,0):$ ions
\[ \alpha = \beta = 5 \times 10^{-1}, B_0 = 3.4 \text{T}, \quad ^3\text{He}, \quad T = 5 \text{eV}, \quad v_{x,0} = v_{y,0} = v_{z,0} = v_t / 3. \]
x $10^{-3}$

| Density $[\text{m}^{-3}]$ | $|E|=18.5\text{keV/m}$ | $|E|=15\text{keV/m}$ | $|E|=9.6\text{keV/m}$ | $|E|=0.6\text{keV/m}$ |
|--------------------------|------------------------|------------------------|------------------------|------------------------|
| $e^-$                    | D                      | H                      | $e^-$                  | D                      | H                      |
| $|E|=18.5\text{keV/m}$   | $|E|=15\text{keV/m}$   | $|E|=9.6\text{keV/m}$  | $|E|=0.6\text{keV/m}$  | $|E|=18.5\text{keV/m}$ | $|E|=15\text{keV/m}$   | $|E|=9.6\text{keV/m}$  | $|E|=0.6\text{keV/m}$  |