Magneto-Convective Instabilities in Horizontal Cavities

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Magneto-convective instabilities in horizontal cavities

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Abstract
A linear stability analysis is performed to investigate the onset of convective motions in a flat cavity filled with liquid metal. A volumetric heat source is uniformly distributed in the fluid and a horizontal magnetic field is imposed. Walls perpendicular to the magnetic field are thermally insulating, the top wall is isothermal and the bottom adiabatic. When a magnetic field is applied electromagnetic forces tend to transform 3D convective flow structures into quasi-2D rolls aligned with the magnetic field. By integrating 3D governing equations along magnetic field lines a quasi-2D mathematical model has been derived. A dissipation term in the 2D equations accounts for 3D viscous effects in boundary layers at Hartmann walls perpendicular to the magnetic field. The influence of various parameters on flow stability is investigated. The flow is stabilized by increasing the magnetic field intensity or the electric conductance of Hartmann walls and by reducing the aspect ratio of the cavity. Numerical simulations are performed to support the analytical results and to describe the main convective flow patterns in the non-linear regime.

Keywords: convection, internal heat source, magnetic field, linear stability

1. Introduction
Thermal convective motions produced by internal heat sources of constant volumetric density in a horizontal liquid metal fluid layer represent a fundamental fluid dynamic problem that is also of interest for engineering applications such as in nuclear fusion reactors. Here a hot plasma of deuterium and tritium is confined in a toroidal vacuum chamber by means of a strong magnetic field. Liquid metals that circulate in the so-called fusion blanket are used both to generate the fuel component tritium and to extract
the generated heat [1]. Most of the nuclear power is deposited in the liq-
uid metal leading to significantly non-uniform thermal conditions that result
in complex convective flow patterns. The latter ones are affected by the
intense magnetic field that confines the fusion plasma. If the liquid metal
forced flow is weak, buoyancy driven motions become important and they
can finally determine flow distribution in fusion blankets [2] [3].

The problem of natural convection driven by a temperature difference
across a horizontal fluid layer, the so-called Rayleigh-Bénard convection, has
been extensively analyzed due to its importance in crystal growth technol-
ogy. When the fluid is heated from below it remains motionless till the
vertical temperature gradient exceeds a critical value. Beyond this thresh-
old the equilibrium state loses its stability and thermal convection sets in.
The application of a magnetic field can stabilize the motion [4]. Studies of
magnetohydrodynamic (MHD) Bénard-convection can be found in [5] and
references therein.

Convective motions can be also driven by heat sources distributed in a
fluid. The steady laminar hydrodynamic convection in an infinite horizontal
fluid layer confined between an isothermal upper plate and an adiabatic lower
plate has been studied by different authors [6] [7]. This configuration differs
from the Bénard-convection problem since the temperature boundary condi-
tions are asymmetric and the vertical temperature profile in the motionless
equilibrium steady state is parabolic rather than linear. First experimental
studies of instabilities in a horizontal layer of fluid heated uniformly and
cooled from above are described in [8]. Roberts [9] carried out a stability
analysis showing that convective motions in an internally heated fluid layer
occur at a critical Rayleigh number $Ra_c \approx 2772$ in the form of marginally
stable rolls. Thirly [10] performed a numerical analysis and determined the
parameters at which polygonal (rectangular and hexagonal) cells occur.

The presence of a magnetic field creates anisotropy in the flow distribution and can have a dual effect on flow instabilities. On the one hand three
dimensional fluctuations are removed on a fast time scale by Joule dissipa-
tion leading to inhibition of convective heat transfer. On other hand MHD
phenomena can intensify vortices aligned with the magnetic field [11]. When
a strong magnetic field is applied the intense electromagnetic Lorentz forces
tend to elongate vortices along magnetic field lines and force the fluid to
move in planes perpendicular to the field, while motion along magnetic field
lines is damped [12]. This leads to a quasi two dimensional (Q2D) MHD
flow where dissipation losses, due to Joule and viscous effects, are localized
in the thin Hartmann layers along walls perpendicular to the magnetic
field. An explanation of dissipative effects in Hartmann layers in MHD duct flows
can be found in [13] [14]. Q2D models consist in reducing the basic govern-
ing equations to a two-dimensional problem by analytical integration along
magnetic field lines. 3D MHD effects are modelled by a term that accounts
for viscous and Joule’s dissipation in Hartmann layers. These numerical
approaches have been employed to predict various MHD problems such as
shear flow instabilities [13], turbulent MHD flows [15], buoyant flows [16]
[17] [18] as well as for interpretation of experimental data [19] [5] and to
investigate problems related to fusion blankets [20] where intense magnetic
fields are present. In [21] it is shown that in an effective Q2D model the
third component of velocity can be taken into account as higher order term
in the asymptotic expansion of variables. Deviations from the initial Q2D
model are as small as \( O(Ha^{-2}) \) and therefore negligible for high Hartmann
numbers.

In the present study we deal with the stability of liquid metal flows in
cavities where a volumetric thermal source is homogeneously distributed in
the fluid in the presence of an applied horizontal magnetic field. The aim is
investigating the influence of various parameters on the onset of instabilities
and identifying the main convective patterns. The geometry chosen for this
study is the same as used by Roberts [9]. In §2 MHD equations and scaling
quantities are introduced. Equations describing the Q2D MHD convective
flow are derived in §3. A stability analysis based on linear disturbance equa-
tions is performed to determine neutral stability depending on the intensity
of the applied heat source and on the strength of the imposed magnetic field
(§4, §5). Results from numerical simulations are discussed in §6.

2. Problem formulation and governing equations

Let us consider an electrically conducting low Prandtl number fluid, such
as a liquid metal, filling a horizontal shallow cavity. A volumetric heat source
\( q \) is uniformly distributed in the fluid. A constant magnetic field \( B_0 \hat{z} \) is
imposed. The top wall at \( y = H \) is isothermal, \( T = T_0 \), the bottom at
\( y = 0 \) and the Hartmann walls at \( z = \pm A \) are adiabatic. In \( x- \) direction the
channel is assumed to be infinitely long. Geometry and coordinate system
are shown in Fig.1.

Changes of density due to temperature variations are described by the
Boussinesq approximation, which states that the fluid density is a linear func-
tion of temperature in the gravitational body force term, \( \rho [1 - \beta (T - T_{ref})] \mathbf{g} \). Here \( \rho \) is the density at the reference temperature \( T_{ref} \), \( \beta \) the volumetric thermal expansion coefficient and \( \mathbf{g} = -g\hat{\mathbf{y}} \) the gravitational acceleration.

Non-dimensional equations governing the problem are those accounting for a balance of momentum, conservation of mass and charge and Ohm's law:

\[
\frac{1}{Pr} \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \nabla^2 \mathbf{v} + Ra T \hat{y} + Ha^2 (\mathbf{j} \times \mathbf{B}), \tag{1}
\]

\[
\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{j} = 0, \tag{2}
\]

\[
\mathbf{j} = -\nabla \phi + \mathbf{v} \times \mathbf{B}. \tag{3}
\]

The electric potential \( \phi \) is obtained by solving a Poisson equation that ensures charge conservation:

\[
\nabla^2 \phi = \nabla \cdot (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{v}). \tag{4}
\]

The temperature distribution is given by the energy balance equation

\[
\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla)T = \nabla^2 T + 1. \tag{5}
\]

The volumetric internal heat source \( q \) is normalized to 1 by defining the characteristic temperature difference as \( \Delta T = qH^2/\lambda \). Fluid properties such as thermal diffusivity \( \alpha = \lambda/(\rho c_p) \), thermal conductivity \( \lambda \), specific heat \( c_p \), kinematic viscosity \( \nu \) and electrical conductivity \( \sigma \) are assumed to be constant in the temperature range considered. In (1)-(5) the dimensionless variables \( \mathbf{v}, t, \mathbf{j}, \phi, p \) and \( \mathbf{B} \) are obtained by scaling velocity, time, electric current density, electric potential, pressure and magnetic field by the reference quantities.
\( v_0 = \alpha / H, \quad H^2 / \alpha, \quad \sigma v_0 B_0, \quad v_0 B_0 H, \quad \rho v_0^3 Pr \) and \( B_0 \), respectively. The length scale \( H \) for the problem is the distance between horizontal walls. The non-dimensional temperature \( T \) is given by \((T^* - T_{ref}) / \Delta T\), where \( T^* \) is the local dimensional temperature. The dimensionless parameters in (1) are the Prandtl number \( Pr \), the Rayleigh number \( Ra \) and the Hartmann number \( Ha \):

\[
Pr = \frac{\nu}{\alpha}, \quad Ra = \frac{g \beta \Delta T H^3}{\nu \alpha} = \frac{g \beta q H^5 \rho c_p}{\nu \lambda^2}, \quad Ha = B_0 H \sqrt{\frac{\sigma}{\rho \nu}}. \tag{6}
\]

The Rayleigh number represents the intensity of the applied heating. The Hartmann number gives a non-dimensional measure for the strength of the imposed magnetic field.

In classical Bénard-convection the strength of convection is quantified by the Nusselt number \( Nu \) that can be regarded as the dimensionless temperature gradient at the wall. When a uniform heat source is distributed in a fluid bounded by a thermally insulating and a perfectly conducting plate, the dimensionless average heat flux per unit area of the upper isothermal wall, i.e. \( Nu \), is unity by definition. Therefore we define a modified Nusselt number \( M \) as the ratio between the mean temperature differences across the fluid layer without and with convective motion. It describes the ratio of the mean temperature required to transport the heat flux in absence of convection to the actual one \( \bar{T} \) when convective transfer is present [9] [10] [22]:

\[
M = \frac{\int_V T_{\text{cond}} dV}{\int_V T_{\text{conv}} dV} = \frac{1}{3\bar{T}}. \tag{7}
\]

3. Two-dimensional model

Two dimensional mathematical models for the description of magnetohydrodynamic flows have been developed by several authors [14] [13] [5].

By applying the curl operator to the momentum equation (1) the pressure is eliminated and we obtain an equation for the vorticity \( \omega = \nabla \times \mathbf{v} \). For \( Pr Ha^2 \gg 1 \) [5], \( \mathbf{B} = \hat{z} \) and boundary conditions \( \mathbf{v} = 0 \) and \( \partial_z T = 0 \) at \( z = \pm \Lambda / H \equiv \pm a \) the flow is characterized by a Q2D velocity \( \mathbf{v} = (u, v, 0) \) and the \( z \)-component of vorticity is given by \( \omega_z \equiv \omega = \partial_x v - \partial_y u \), where \( u, v, \omega \) may depend on \((x, y, z)\). This leads to an equation for the \( z \)-component of the vorticity:

\[
\frac{1}{Pr} \left( \partial_t \omega + (u \partial_x \omega + v \partial_y \omega) \right) = \nabla^2 \omega + Ha^2 \partial_z j_z + Ra \partial_z T. \tag{8}
\]
In 2D flows velocity and vorticity can be expressed by a separation ansatz
\[ u = \hat{u}(t, x, y) f(z), \quad v = \hat{v}(t, x, y) f(z), \quad \omega = \hat{\omega}(t, x, y) f(z). \quad (9) \]
Quantities indicated with a hat represent amplitudes of velocity and vorticity components and \( f(z) \) is a shape function that describes the variation of the variables along magnetic field lines. This function has to satisfy the no slip-condition at the Hartmann wall, i.e. \( f(z = a) = 0 \), and a symmetry condition \( \partial_z f(z = 0) = 0 \). Expressions (9) are substituted in (8). The final equation
\[ \frac{1}{Pr} \left( \partial_t \hat{\omega} + \hat{u} \partial_x \hat{\omega} + \hat{v} \partial_y \hat{\omega} \right) = \nabla_{xy}^2 \hat{\omega} - \frac{1}{\tau} \hat{\omega} + Ra \partial_x T, \quad (10) \]
with
\[ \frac{1}{\tau} = \frac{Ha}{a} + \frac{cHa^2}{a + c}, \quad (11) \]
is obtained by integrating vorticity equation (8) along magnetic field direction, by applying the thin wall condition that describes charge conservation in electrically conducting walls [23]
\[ j_{z,w} = -c \nabla_{xy}^2 \hat{\phi}_w, \quad (12) \]
and by using (4). In (12) \( \nabla_{xy} \) is the projection of the gradient operator on the plane of the Hartmann wall and \( c = \sigma_w t_{w}/(\sigma H) \) is the conductance parameter. The latter one describes the ratio of the electrical conductance of the Hartmann wall with thickness \( t_w \) and electrical conductivity \( \sigma_w \) to the one of the fluid. In (10) on the right hand side the first dissipation term \( \nabla_{xy}^2 \hat{\omega} \) describes viscous losses caused by gradients of vorticity in a plane perpendicular to the magnetic field and \( -\hat{\omega}/\tau \) represents viscous dissipation in Hartmann layers and Joule dissipation in layers and in the thin electrically conducting walls. The dissipation factor \( \tau^{-1} \) is related to a typical decay time of vorticity due to Joule heating and viscous losses [13]. It can be observed that for electrically insulating Hartmann walls \( (c = 0) \tau^{-1} \rightarrow Ha/a \) and for perfectly conducting walls \( (c = \infty) \tau^{-1} \rightarrow Ha^2 \). This means that in channels with highly electrically conducting walls a strong and rapid damping occurs.

4. Linear stability analysis

The stability of the motionless basic steady state solution characterized by \( \mathbf{v}_0 = (0, 0, 0), \omega = 0 \) and a parabolic temperature distribution along the
vertical coordinate $y$, $T_0 = 1/2(1 - y^2)$, is investigated. The aim is identifying critical Rayleigh number $Ra_c$ and critical wave number $k_c$ for the onset of convective instabilities in the geometry shown in Fig.1. When the internal heat source, i.e. the Rayleigh number $Ra$, is large enough, the steady state loses its stability due to increased buoyancy forces that are not balanced anymore by viscous effects.

In order to derive disturbance equations, temperature, velocity and vorticity are decomposed as the sum of a basic state denoted by the subscript $0$ and perturbations indicated by prime, e.g. $T = T_0 + \varepsilon T'$. We introduce these expressions in (10) and (5) and by neglecting terms of $O(\varepsilon^2)$ and smaller we obtain the linearized equations

$$
\frac{1}{Pr} \partial_t \omega' = \nabla_{xy}^2 \omega' - \frac{1}{\tau} \omega' + Ra \partial_x T',
$$

$$
\partial_t T' - u'y = \nabla_{xy}^2 T'.
$$

The stability is determined by solving the resulting eigenvalue problem [5]. In order to satisfy mass conservation, $\nabla \cdot \mathbf{v} = 0$, a streamfunction $\psi'(x, y)$ is introduced such that $\mathbf{v} = \nabla \times (\psi' \mathbf{\hat{z}})$ and $\omega' = -\nabla^2 \psi'$. We expand perturbations in normal modes as $\omega' = \Omega(y)e^{st + ikx}$, $\psi' = \Psi(y)e^{st + ikx}$, $T' = i\Theta(y)e^{st + ikx}$, where $k$ is a real horizontal wavenumber, $s$ is the temporal rate of growth of the perturbations and $\Omega$, $\Theta$, $\Psi$ are amplitude functions. After some mathematical work the equations describing the stability of the problem become:

$$
\left( D^2 - k^2 - \frac{1}{\tau} \right) \Omega - kRa\Theta = \frac{s}{Pr} \Omega,
$$

$$
\left( D^2 - k^2 \right) \Theta - ky \Psi = s\Theta,
$$

$$
\left( D^2 - k^2 \right) \Psi + \Omega = 0,
$$

where $D^2 = \partial^2 / \partial y^2$. Equations (15)-(17) with boundary conditions $\Psi = 0$, $\partial_y \Theta = 0$ at $y = 0$ and $\Psi = 0$, $\Theta = 0$ at $y = 1$ constitute an eigenvalue problem.

A numerical procedure has been implemented in Matlab where finite difference techniques are used for the solution of the eigenvalue problem (15)-(17). For a given wave number $k$ the Rayleigh number $Ra$ is varied until a value is reached for which an eigenvalue $s$ exists with a real part equal to zero, i.e. until the solution reaches the stability limit. This couple $(k, Ra)$ represents a point on the neutral curve (see Fig.2). Instability sets in first for $Ra_c = \min(Ra)$ at the corresponding critical wave number $k_c$. Once we
have the eigenvalue we can determine the corresponding eigenvector to draw streamlines and temperature contours to visualize the flow pattern at the onset of instability.

5. Results from the linear stability analysis

In order to verify the accuracy of the numerical procedure implemented in Matlab, we investigate first (§5.1) the hydrodynamic flow ($Ha = 0$) as considered by Roberts [9]. In a second step a uniform horizontal magnetic field is imposed. For the discussion of the results one should keep in mind that the Q2D model is valid only for sufficiently large Hartmann numbers $Ha$. However, we can recover the hydrodynamic solution both by taking $Ha = 0$ or $a \to \infty$, since under these conditions the magnetohydrodynamic dissipation factor $\tau^{-1}$ (11) vanishes, namely dissipation in Hartmann layers plays no role anymore.

5.1. Hydrodynamic case

Figure 2: Neutral stability curve for the hydrodynamic flow ($Ha = 0$) showing the marginal Rayleigh number $Ra$ as a function of the wavenumber $k$. The minimum of the curve gives the critical values of $Ra$ and $k$ ($Ra_c = 2772$, $k_c = 2.63$). Contours of the streamfunction are visualized at the onset of instability, showing counter rotating cells.

Figure 2 shows the neutral stability curve for the hydrodynamic case. As the control parameter $Ra$ increases the base state becomes linearly unstable at $Ra = Ra_c$ with respect to perturbations with horizontal wavenumber
In Fig. 2, contours of the streamfunction are depicted showing two counter-rotating convective cells over a wavelength, as they appear at the instability threshold. Results agree very well with those from the linear stability analysis performed by Roberts [9]. A mesh sensitivity study has been performed to guarantee grid independence of the results. The critical Rayleigh number and the corresponding wavenumber have been predicted by progressively increasing the number of grid points \( n \) in the vertical direction between the plates. For \( n \geq 200 \), deviation from results in [9] is of the order of 0.1%. Results described in this section are obtained for \( n = 400 \).

5.2. Influence of a horizontal magnetic field

![Figure 3: Critical Rayleigh number \( Ra_c \) and critical wave number \( k_c \) as a function of the dissipation factor \( 1/\tau \).](image)

In this section we investigate the influence on flow stability of parameters such as the strength of the imposed magnetic field, given in terms of Hartmann number \( Ha \), the electrical conductivity of Hartmann walls \( c \) and the aspect ratio \( a = A/H \) of the cavity. The behavior of a liquid metal layer in an electrically insulating cavity \( (c = 0) \) with aspect ratio \( a = 2 \) is taken as reference case. Results can be summarized in a single diagram (Fig. 3) where critical Rayleigh number and corresponding wavenumber are plotted as a function of the dissipation factor \( \tau^{-1} \) (11). For weak dissipation \( (\tau^{-1} \ll 1) \), \( Ra_c \) and \( k_c \) tend to constant values (hydrodynamic conditions) and for \( \tau^{-1} \gtrsim 10^2 \), \( Ra_c \propto \tau^{-1} \).
Figure 4: Neutral stability curves for flow with $c = 0$, $a = 2$ and various Hartmann numbers $Ha$. Results show the stabilizing effect of an increasing magnetic field on the flow.

In Fig.4 neutral stability curves are depicted for various Hartmann numbers $Ha$. The curve for the hydrodynamic case ($Ha = 0$) is shown for comparison. It can be seen that, as expected [4] [5], the magnetic field has a stabilizing effect on the flow, namely by increasing the Hartmann number the onset of convection occurs at higher values of the Rayleigh number $Ra$. In Fig.5 the variation of $Ra_c$ and $k_c$ as a function of the Hartmann number is displayed. For $Ha \gg 1$ the critical Rayleigh number varies linearly with $Ha$, $Ra_c \propto Ha$, while $k_c$ varies in a very small range between 2.4 and 2.65. For small Hartmann numbers the Q2D model is not valid any longer, but the hydrodynamic solution is recovered by taking $Ha = 0$, since for such flow conditions the Hartmann braking is absent ($\tau^{-1} = 0$).

A shift of the onset of convective motion to higher $Ra$, namely a stabilization of the flow, is also achieved by reducing the aspect ratio of the cavity. This can be seen in Fig.6 where the critical Rayleigh number $Ra_c$ and the corresponding wavenumber $k_c$ are shown as a function of the magnetic field strength for various aspect ratios $a$. By decreasing $a$ the coefficient $\tau^{-1}$ (11) of the damping term of the vorticity increases. For constant $Ha$ the thickness of the Hartmann layer ($\delta_Ha \approx Ha^{-1}$) in which the braking of the flow takes place remains constant but by reducing $a$ the volume of the cavity becomes smaller. Therefore the role of the Hartmann braking increases and as a result $Ra_c$ at which the basic flow state loses stability becomes larger.
The hydrodynamic case represents practically a limiting curve for $a \to \infty$, i.e. a flow between two parallel infinitely extended horizontal plates. Critical Rayleigh number and critical wave number for hydrodynamic conditions agree well with [9]. Results in Fig.6 show that for sufficiently intense magnetic field $Ra_c$ increases linearly with $Ha$. This dependence becomes clear by observing the governing equation (10) derived as part of the 2D model. For strong magnetic fields a balance establishes between buoyancy ($Ra \partial_x T$) and electromagnetic dissipation ($\tau^{-1} \dot{\omega}$), namely $Ra_c \propto \tau^{-1}$ and for electrically insulating cavities ($c = 0$) $Ra_c \propto Ha/a$.

The influence of the conductance parameter $c$ on flow stability is displayed in Fig.7. For $c > 0$ the inverse of the decay time approaches $\tau^{-1} \approx cHa^2/(a+c)$ according to (11) and for perfectly conducting walls, i.e. $c = \infty$, $Ra_c \propto \tau^{-1} \to Ha^2$. This means that in ducts with better electrically conducting Hartmann walls perturbations are more rapidly suppressed than in an insulating channel and the flow is more stable. The total current density increases and hence the damping is stronger.
Figure 5: Critical Rayleigh number $Ra_c$ and corresponding critical wave number $k_c$ as a function of the Hartmann number $Ha$, for $c = 0$ and $a = 2$.

Figure 6: Critical Rayleigh number $Ra_c$ and critical wavenumber $k_c$ as a function of $Ha$ for magneto-convective flows in a cavity with $c = 0$ and various aspect ratios $a$. 

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Figure 7: Critical Rayleigh number and critical wavenumber as a function of $Ha$ for different wall conductance parameters $c$ and $a = 2$. For $Ha \rightarrow 0$ $Ra_c \approx 2770$ and $k_c = 2.63$ in agreement with [9]. For $c = 0$ we observe that $Ra_c \approx Ha/a$ and for $c \rightarrow \infty$ $Ra_c \propto Ha^2$. 
6. Numerical simulations

In order to get insight into the nonlinear behavior of the flow after the onset of convection, numerical simulations are carried out by a finite volume solver developed in the open source package OpenFOAM. The solver has been fully validated against analytical and asymptotic solutions for forced and buoyant MHD flows [24] [25]. The following model equation for velocity components perpendicular to the magnetic field, \( \mathbf{u} = (u, v, 0) \), is solved and its curl yields (10):

\[
\frac{1}{Pr} (\partial_t \mathbf{u} + (\mathbf{u} \nabla_x) \mathbf{u}) = -\nabla_{xy} p + \nabla^2_{xy} \mathbf{u} - \frac{1}{\tau} \mathbf{u} + Ra T \mathbf{y}. \tag{18}
\]

This equation is solved by applying so called "empty" boundary conditions in magnetic field direction, which assumes no changes of variables in the core along magnetic field lines. The electric current density conservative scheme as proposed by [26] is employed for solving the potential equation (4).

6.1. Onset of instability and evolution of flow pattern

Let us consider the flow in an electrically insulating cavity \((c = 0)\) with aspect ratio \(a = 2\) at a constant Hartmann number \(Ha = 200\). For such conditions the stability analysis predicts a critical Rayleigh number \(Ra_c \approx 9822\) for the onset of convective motion. For \(Ra < Ra_c\) conductive heat transport between the plates represents the only stable solution. For \(Ra > Ra_c\) convection sets in and the flow reaches a new solution characterized by convective rolls as depicted by means of streamlines in Fig.8 for the flow at \(Ra = 10000\). Perturbations in the temperature field have the form of harmonic waves, as visible in Fig.8(a), where isotherms are depicted on the vertical symmetry plane \(z = 0\). Although the electric potential \(\phi\) doesn’t appear in the equations of the Q2D model, we calculate this variable according to (4), since it can serve as approximate streamfunction for the flow (see potential contours and streamfunction isolines in Fig.8(b)) [27] and it can be directly compared with results from 3D calculations. Contours of vertical velocity are plotted in Fig.8(c). Profiles of the vertical velocity along the central horizontal line of the cavity at \(y = 0.5\) and \(z = 0\) are shown in Fig.9 for increasing Rayleigh numbers. For small supercritical \(Ra\) mainly harmonic functions give a contribution when convective motion sets in. By rising \(Ra\) additional modes and higher wave numbers appear due to nonlinear interactions leading to more complex velocity profiles.
In Fig. 10 contours of instantaneous vertical velocity component \( v \) are plotted on the vertical symmetry plane \( z = 0 \) of the cavity for increasing \( Ra \). There is a transition from a symmetric roll pattern at low supercritical Rayleigh numbers to a butterfly-like pattern for larger \( Ra \). This is caused by the progressive occurrence of additional modes (compare Fig. 9). The modifications of the velocity field and the increasing heat transfer resulting from a larger applied volumetric heating (\( Ra \)) are reflected also by the temperature distribution. Contours of instantaneous temperature are visualized in Fig. 11 for the flow at \( Ha = 200 \) and five \( Ra \). In the temperature field there is a transition from a harmonic wave perturbation to a mushroom pattern.

In Fig. 12 the temperature, averaged in time and along the coordinate \( x \), is plotted as a function of \( y \) for various \( Ra \). The stable motionless state is characterized by a parabolic temperature profile and by increasing the internal volumetric heating the temperature distribution deforms due to the onset of convective motions. In the selected scale for the temperature (see §2) \( \Delta T \) is proportional to the volumetric heat source and therefore it increases with \( Ra \) so that the normalized temperature reduces when convection is present. For \( Ra > 150000 \) the flow becomes time dependent and velocity and temperature patterns oscillate in time loosing the regular structure typical of stationary solutions (see flow at \( Ra = 300000 \) in Figs. 10-11).
Figure 9: Vertical velocity component along the central horizontal line of the cavity at $y = 0.5$ and $z = 0$ for the flow at $Ha = 200$ and various $Ra$. By rising $Ra$ additional modes appear due to nonlinear interactions.

Figure 10: Contours of vertical velocity $v$ and streamfunction isolines for $Ha = 200$, $c = 0$, $a = 2$ and various $Ra$. The case $Ra = 300000$ is time-dependent.
Figure 11: Contours of temperature for flows at $Ha = 200$, $c = 0$, $a = 2$ and various $Ra$. Results up to $Ra = 90000$ are stationary. The case $Ra = 300000$ is time-dependent.

Figure 12: Temporally and axially averaged temperature along the vertical coordinate $y$ for various Rayleigh numbers.
6.2. Solutions with different number of convective cells

For the considered cavity with a fixed size in $x$—direction equal to 4 wavelengths, i.e. $l_x/H = 8\pi/k_c$, numerical simulations show for the same Rayleigh number the coexistence of solutions with a different number of convective cells. When the number of cells increases the flow becomes more regular due to the larger dissipation and owing to the more intense convective heat transfer. Examples are shown in Figs.13-14 for the flow at $Ra = 300000$. Here contours of instantaneous vertical velocity and temperature are displayed. Solutions with 8, 10 and 12 rolls are time dependent, while the one with 16 cells is stationary. The asymmetry in the velocity and temperature patterns reduces when the number of cells increases.

![Figure 13: Flow at Ha = 200 and Ra = 300000. Contours of vertical velocity are shown for four different solutions with increasing number of convective cells.](image)

The intensity of the convective heat transfer is quantified by the dimensionless number $M$ (7). Its variation as a function of $Ra$ gives the bifurcation diagram shown in Fig.15 for $Ha = 200$. The numerically predicted critical Rayleigh number for the onset of convection, $Ra_c = Ra(M = 1)$, agrees very well with the one obtained by the stability analysis. By increasing the
Figure 14: Flow at $Ha = 200$ and $Ra = 300000$. Contours of temperature are shown for four different solutions with increasing number of convective cells.

Figure 15: Modified Nusselt number $M$ that quantifies the strength of convective motion as a function of the Rayleigh number $Ra$.
Rayleigh number $M$ becomes larger, i.e. the convective heat transfer intensifies. Various solutions with a different number of convective rolls coexist. Due to the selected length of the cavity the most stable solution is the one with eight convective cells. By boosting the internal heating in the liquid metal, solutions with more rolls appear that become stable for sufficiently large $Ra$. When a larger number of convective cells is present the heat transfer becomes stronger. This effect can be better appreciated by considering the data in Fig.16 where contours of normalized temperature $T$ and isolines of electric potential are depicted on the vertical symmetry plane of the cavity for the flow at $Ha = 200$ and $Ra = 70000$ by using the same color scale. When the number of rolls increases the mean temperature reduces due to enhanced convective heat transfer and stronger motion. In Fig.17 the axial distribution of the vertical component of velocity is displayed for the flow at $Ra = 70000$ for solutions with 8 and 16 convective cells. When 8 rolls are present the velocity profile shows the contribution of additional modes.

Figure 16: Contours of temperature and electric potential isolines (non-dimensional distance is about 0.9) showing different numbers of rolls for Q2D MHD flow at $Ha = 200$ and $Ra = 70000$.

6.3. Transition from an unstable to a stable configuration

Another interesting phenomenon is the transition from an unstable configuration with 16 rolls to a stable one with 8 cells. In Fig.18 snapshots are displayed showing the evolution of electric potential distribution and streamlines when passing from 16 to 8 cells for the flow at $Ha = 200$ and
Figure 17: Vertical velocity component along the central horizontal line of the cavity at \( y = 0.5 \) and \( z = 0 \) for the flow at \( Ha = 200 \) and \( Ra = 70000 \).

\( Ra = 40000 \). Pairs of convective rolls become smaller and then they disappear. The selected instants are indicated by means of red stars in Fig.19, where the modified Nusselt number \( M \) is plotted as a function of the time to show the transition from the unstable to the stable convective solution. In the sub-plots in Fig.19 contours of electric potential and temperature are depicted for comparison for the flows at \( t = 12.8 \) and \( t = 30 \).

7. Conclusions

The influence of a horizontal magnetic field on the stability of a fluid layer in which a volumetric heat source is uniformly distributed has been investigated. A linear stability analysis has been performed based on a Q2D model. When convective motion sets in perturbations are described by stationary harmonic functions along the horizontal \( x \)-direction perpendicular to magnetic field lines; the temperature is characterized by a wavy regular pattern and the flow is organized in rolls aligned with the magnetic field. By increasing \( Ra \) higher modes appear due to nonlinear interactions and a transition is observed towards a mushroom pattern for the temperature field and a butterfly configuration for the vertical component of velocity. For the largest heat sources considered, the solution becomes time dependent and exhibits asymmetric convective patterns. The motionless basic state is stabilized by increasing the magnetic field intensity (\( Ha \)) or the electric con-
ductance parameter $c$ of Hartmann walls and by reducing the aspect ratio $a$ of the cavity. For supercritical conditions solutions with different wavenumbers coexist in the considered computational domain. When the number of convective rolls increases, the heat transfer intensifies as indicated by a larger modified Nusselt number $M$.

Preliminary 3D simulations show for sufficiently small $Ra$ good agreement with results obtained with Q2D equations. For large $Ra$ or smaller $Ha$ perturbations may lose their 2D character and start varying along the magnetic field direction. This transition from a 2D to a 3D flow is the focus of ongoing numerical investigations with the aim of defining the range of validity of the Q2D model. It should be also noted that in the Q2D model the influence of the layers that develop along walls parallel to the magnetic field, the so-called side walls, is not fully taken into account. Therefore the effect on flow stability of those layers and of the electric conductivity of the adjacent walls should be clarified.
Figure 19: Dimensionless number $M$ as a function of the simulation time for the flow at $Ha = 200$ and $Ra = 40000$. The time is scaled by $H^2/\alpha = H/v_0$. The jump indicates the transition from an unstable solution with 16 convective cells to a stable one with 8 rolls. Contours of electric potential (above) and temperature (below) are shown for comparison for flows at $t = 12.8$ and $t = 30$. Red stars on the curve indicate the selected instants at which snapshots are displayed in Fig.18.

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References


