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Contents

Second order Gyrokinetic theory for Particle-In-Cell codes.

Natalia Tronko[1], Alberto Bottino[1] and Eric Sonnendrücker[1]

[1] *Max-Planck-Institut für Plasmaphysik, 85748 Garching, Germany*

The main idea of Gyrokinetic dynamical reduction consists in systematical removing of fastest scale of motion (the gyro motion) from plasma's dynamics, resulting in a considerable model simplification and gain of computing time.

Gyrokinetic Maxwell-Vlasov system is broadly implemented in nowadays numerical experiments for modeling strongly magnetized plasma (both laboratory and astrophysical). Different versions of reduced set of equations exist depending on the construction of the Gyrokinetic reduction procedure and approximations assumed while their derivation.

The purpose of this paper is to explicitly show the connection between the general second order gyrokinetic Maxwell-Vlasov system issued from the Modern Gyrokinetic theory derivation and the model currently implemented in global electromagnetic Particle in Cell code ORB5.

Strictly necessary information about the Modern Gyrokinetic formalism is given together with the consistent derivation of the gyrokinetic Maxwell-Vlasov equations from the first principle of dynamics. Variational formulation of dynamics is also used for simultaneously obtaining the corresponding exact energy conservation law. The result of that explicit derivation is used for verification of the energy conservation diagnostics currently implemented in ORB5 code.

This work subscribes into the context of codes verification project VeriGyro currently run in IPP Max Planck Institut in collaboration with others European institutions.

I. INTRODUCTION

For more than five decades, magnetized plasmas have been investigated in order to achieve self-sustained nuclear reaction process in fusion devices. The complexity of these systems urges numerical simulations to understand the dynamical behavior of the plasma. However, numerical simulations rely on theoretical models which have to reach a compromise between an accurate description of the dynamics and a restrained number of numerical operations to keep simulations tractable in current computing facilities.

Gyrokinetic theory aims at this compromise by taking advantage of the specific motion of the plasma. More precisely, the presence of a strong background magnetic field in such devices makes possible the separation of different scales of motion. The main idea is to separate the fast motion

of charged particles around magnetic field lines (referred to as gyromotion) from their slower drift motion, in order to reduce the number of dynamical variables needed to describe the dynamics. The cyclotron frequency $\Omega = eB/mc$, where e and m are respectively the charge and mass of particles, B is the magnetic field amplitude and c is the speed of light, sets the scale of the gyromotion. Mathematical tools and approximations allowing for the splitting out of this fast scale define a particular gyrokinetic dynamical reduction.

The gyromotion is described by a fast gyroangle variable θ, to which corresponds a canonically conjugate slowly varying magnetic moment μ, called adiabatic invariant of the system. At the lowest order of approximation $\mu = mv_\perp^2/2B$, where $v_\perp$ is the perpendicular velocity of particles respective to background magnetic field lines. In early works [7], an iterative gyro-averaging procedure has been used in order to remove the θ-dependency directly from the Vlasov equation. Such a procedure allowed for the derivation of non-linear gyrokinetic equations. However, the major issue was the impossibility to obtain an energy-conserving model from this procedure.

The Modern Gyrokinetic theory [5, 13] makes use of differential geometry (perturbative Lie-transformation techniques) to build up a new set of phase-space variables, such that the fast gyroangle variable θ becomes uncoupled from the description of particle's motion and the corresponding moment has trivial dynamics $\dot{\mu} = 0$. Therefore, the particle phase space is reduced from 6 dimensions to 4+1 dimensions, which already represents a significant simplification for numerical simulations.

However, one of the main difficulties is then to find a rigorous way for coupling the reduced particle dynamics to those of the dynamical electromagnetic fields induced by particles, in order to obtain a self-consistent description of the reduced system.

Two variational formulations exist, Lagrangian [16] and Eulerian [3], both providing a common framework for the description of Gyrokinetically reduced self-consistent field-particles dynamics allowing for the derivation of energy and momentum conservation laws. In these formulations, particles are described on the reduced phase space, while dynamical fields are still being evaluated at the non reduced space positions. One of the main advantages of the variational formulation is contained in the fact that polarization and magnetization effects arise naturally as a result of the dynamical reduction from coupling with reduced particle's dynamics.

In the Lagrangian formulation, dynamics of particles is represented via the characteristics, from which the Vlasov equation is reconstructed a posteriori. It allows to choose a reduced model for particle's dynamics (i.e. linear polarization approach) and to systematically couple this reduced model to the dynamical electromagnetic fields. However, additional calculation of moments is

required in order to reconstruct energy and momentum densities allowing for the derivation of conservation laws through Noether's theorem [15]. The Lagrangian formulation is the natural framework for Particle-in-Cell (PIC) code discretization [12], [1].

Within the Eulerian approach, particles are represented via the Vlasov distribution function, which is treated as one of the dynamical fields of the theory. This leads to direct derivation of conservation laws via Noether's method and does not require any external moments calculation [6]. This approach allows to proceed with systematic derivation of the reduced Maxwell-Vlasov model by truncating the action functional corresponding to the Gyrokinetic system at the desirable order. In the same time, the Eulerian formulation is well suited to handle the splitting between background and fluctuating quantities. Such a manipulation on Vlasov distribution function is used for the description of instabilities and could be particularly useful to keep ordering consistency within the reduced Vlasov equation.

In this article we derive a second order Maxwell-Vlasov gyrokinetic model of reference from the systematic variational approach, suitable for code verification. We compare the result with the Gyrokinetic equations recently implemented in PIC code ORB5 [11], [2].

We start our derivation by writing an explicit second order expression for the Eulerian action functional presented in [4]. From the first variation of the action functional, we derive the corresponding Maxwell-Vlasov equations of motion. We then derive the energy conservation law thanks to the Noether's method. We finally get the Eulerian second order action principle corresponding to the gyrokinetic model implemented in ORB5 [11] code.

This paper is organized as follows: in section II we explicit strictly necessary for the following model's equations derivation elements of the gyrokinetic reduction procedure for particle dynamics. Section III begins with summarizing the main concepts of Eulerian variational principle for gyrokinetic Maxwell-Vlasov system. We then derive the expression for the full second order Eulerian action and the corresponding reduced equations of motion.

Section IV starts with derivation of corresponding truncated Maxwell-Vlasov equations implemented in ORB5 code from the corresponding Eulerian action functional. The obtained equations are then compared to the result of the Eulerian variational principle implementation from the section IV.

In section VI we are using Noether's method to get the expression for conserved energy from the second order Eulerian action. Then we compare the result of Noether's derivation with quantity implemented for energy conservation diagnostics into the ORB5.

II. GYROKINETIC DYNAMICAL REDUCTION ON PARTICLE'S PHASE SPACE: SOURCES OF POLARISATION AND MAGNETISATION.

In that section we focus our attention on gyrokinetic dynamical reduction procedure for a single particle dynamics in external electromagnetic fields. This is a preliminary step necessary for self-consistently reduced Maxwell-Vlasov model derivation. The main goal consists in exporing the intimate link between the reduced particle's dynamics and polarization/magnetization effects, which will appear lately into the Gyrokinetic Maxwell equations.

Clarifying effects of particle's dynamical reduction constitute an important preliminary step for correct coupling of reduced dynamics with fields and therefore for the gyrokinetic field theory construction.

The idea of gyrokinetic dynamical reduction is tightly related to existence of adiabatic invariant, magnetic moment μ, which in the simplest case of slab magnetic geometry is given by $\mu = mv_\perp^2/2B$. Magnetic moment measures the area disclosed by particle motion while its rotation around magnetic field line. From this geometrical meaning comes the idea of using μ as an action variable canonically conjugated to the fast gyromotion around magnetic field lines.

In a straight uniform magnetic field, μ is an exact invariant due to the fact that while its rotation, particle closes the loop exactly. Effects of magnetic field curvature as well as presence of fluctuating electromagnetic field component destroy that exact conservation: particles do not completely close the loop anymore. However, the averaged over long times magnetic moment is still being conserved $\langle \dot{\mu} \rangle_t = 0$.

The dynamical reduction can be organized in one or in two steps depending on the choice of splitting or not contributions to the fast dynamics from the background and fluctuating fields. Choosing the two step dynamical procedure is helpful for understanding contributions to the polarization and magnetization from the gyrokinetic reduction. To each step corresponds set of new phase space coordinates such that the fast gyro motion is uncoupled from the description of slow particle's drifts; and the dynamics on the reduced phase space is restricted to the surface $\dot{\mu} = 0$.

These new coordinates are constructed as perturbative series of near-identity phase space transformations, i.e. the transformation can be inverted at each step of perturbative procedure.

Before we proceed with detailed description of reduction procedures, we need to discuss small parameters associated with each change of coordinates.

A. Gyrokinetic orderings

At the first step, called the guiding-center transformation, only strong nonuniform background magnetic field effects are taken into account. We relate to that step a small parameter $\epsilon_B = \rho_{th}/L_B$, representing ratio between the thermal ion Larmor radius and the scale L_B on which background magnetic field exhibits important changes. Remark that in early works of Northtrop and Littlejohn, small parameter associated with the guiding-center dynamical reduction appears as a formal parameter, which scales like the inverse of the electric charge: $\epsilon \sim e^{-1}$.

At the second step, called the gyrocenter transformation, the reduced guiding-center system is perturbed by external fluctuating electromagnetic fields. That leads towards mixing of time scales and therefore breaks down the magnetic moment conservation at the order of the amplitude of perturbation. The goal of the gyrocenter transformation is to re-establish time scales separation and conservation of μ for the perturbed system. The small parameter related to that step of dynamical reduction measures relative amplitude of fluctuating fields $\epsilon_\delta = \epsilon_\perp e\delta\phi/T_i$, where $\epsilon_\perp = |\mathbf{k}_\perp \rho_{th}|$; $\delta\phi$ here represents the amplitude of the fluctuating electrostatic potential and T_i is the ion temperature.

For the maximal gyrokinetic ordering consistency, one should consider contributions from each dynamical reduction procedure at the same order: $\epsilon_B \sim \epsilon_\delta$. However, in most of nowadays numerical simulations, contributions from the background magnetic field curvature are pushed at the next order, i.e. $\epsilon_B \ll \epsilon_\delta$, which can be relevant for example for a large aspect ratio simulations. As we will see further on, the same approximation will allow to use the Bessel function J_0 for calculation of the gyro - averaging.

B. Gyrokinetic particle's Lagrangian

Particle's Lagrangian Λ is the central object of dynamical reduction procedure for uncoupled single particle dynamics: it consists of a symplectic part $\mathbf{\Gamma}$ and a Hamiltonian part H:

$$\Lambda = \mathbf{\Gamma} \cdot \dot{\mathbf{X}} - H. \tag{1}$$

The symplectic part provides the information about the symplectic structure on the reduced phase space. The components of the symplectic matrix ω are related to the components of the vector $\mathbf{\Gamma}$ as follows:

$$\omega_{\alpha\beta} = \frac{\partial \Gamma^\beta}{\partial Z^\alpha} - \frac{\partial \Gamma^\alpha}{\partial Z^\beta}, \tag{2}$$

where Z^α represent reduced phase space coordinates, which we will explicitly define in the following subsection.

Together the symplectic structure ω and the Hamiltonian H provide necessary information for derivation of the equations of motion on the reduced particle phase space. Defining the expression for the Poisson matrix as the inverse of the symplectic matrix $\Pi^{\alpha\beta} = \omega^{-1}_{\alpha\beta}$, and the Poisson bracket as $\{F, G\} = \frac{\partial F}{\partial Z^\alpha}\Pi^{\alpha\beta}\frac{\partial G}{\partial Z^\beta}$, we can write the reduced equations of motions $\dot{Z}^\alpha = \{z^\alpha, H\} = \Pi^{\alpha\beta}\frac{\partial H}{\partial Z^\beta}$.

In this work, we will specify the expression for the reduced Poisson bracket and the reduced Hamiltonian H up to the second order of dynamical reduction.

1. Local particle's coordinates

The dynamical reduction is performed in *local* particle's coordinates z^α. For this purpose, one needs to define two vector basis: the static one, which remains its position while particle's rotation around magnetic field line; and the dynamical one, which rotates with particle around the magnetic field line. As a static basis we take the natural Frenet triad, associated with unitary vector $\widehat{\mathbf{b}} = \mathbf{B}/B$ in a direction of background magnetic field. In the perpendicular to the background magnetic field plane we use then the normalized curvature vector $\widehat{\mathbf{b}}_1 = \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\widehat{\mathbf{b}} / \left|\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\widehat{\mathbf{b}}\right|$ and we define third basis vector as: $\widehat{\mathbf{b}}_2 = \widehat{\mathbf{b}} \times \widehat{\mathbf{b}}_1$. The dynamical triad is constructed from the static one as follows: we take $\widehat{\mathbf{b}}$ as its first vector and we define

$$\widehat{\boldsymbol{\rho}} = \widehat{\mathbf{b}}_1 \cos\theta - \widehat{\mathbf{b}}_2 \sin\theta \tag{3}$$

and

$$\widehat{\perp} = -\widehat{\mathbf{b}}_1 \sin\theta - \widehat{\mathbf{b}}_2 \cos\theta. \tag{4}$$

Where θ is the gyrophase angle: $\tan\theta = \left(\widehat{\boldsymbol{\rho}} \cdot \widehat{\mathbf{b}}_2\right) / \left(\widehat{\perp} \cdot \widehat{\mathbf{b}}_2\right)$, is now one of the coordinates into the perpendicular plane.

We are now ready to define the local particle coordinates: $z^\alpha = (\mathbf{x}, v_\parallel, \mathbf{v}_\perp) \to (\mathbf{x}, v_\parallel, \mu, \theta)$ with performing the following coordinate decomposition for particle velocity:

$$\mathbf{v} = v_\parallel \widehat{\mathbf{b}} + \mathbf{v}_\perp \equiv v_\parallel \widehat{\mathbf{b}} + (2\mu m B)^{1/2} \widehat{\perp}, \tag{5}$$

here $\mu = \frac{m v_\perp^2}{2B}$ is the magnetic moment of the particle.

The goal of the dynamical reduction procedure is to define a near-identity change of coordinates on particle's phase space, such that the fast dynamics associated with gyromotion is uncoupled

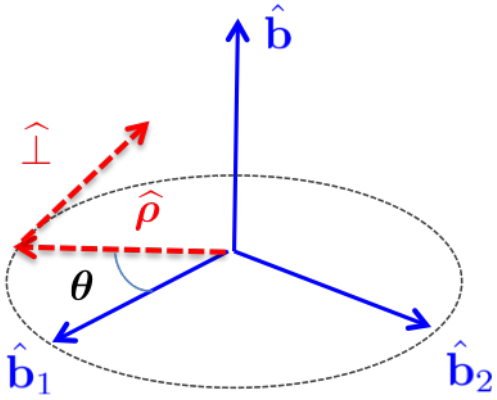

FIG. 1: The Frenet triade, used for definition of local particle coordinates.

from the dynamical description. We define the coordinates on the reduced phase space as follows: $Z^\alpha = \tau_\epsilon z^\alpha$, where τ_ϵ is the near-identity operator defined from the general perturbative procedure. We omit details of this general procedure here, they can be found in [17]. However, it is important to notice that in general case, this operator τ_ϵ is gyrophase, i.e. θ-dependent. In the other words, it means that in order to rewrite dynamics into the variables, which allow to uncouple fast dynamics, one needs to perform a coordinate transformation, which itself depends on this fast variable.

2. Gyrogauge dependence

Before we proceed further on, we need to point out an important geometrical issue: the gyro-gauge dependence of the static basis. Indeed, the static basis in the plane perpendicular to the $\widehat{\mathbf{b}}$ direction is not defined in a unique way. On the Fig. 2 we explain this graphically. The angle ξ makes the difference between two basis vectors $(\widehat{\mathbf{b}}_1, \widehat{\mathbf{b}}_2)$ and $(\widehat{\mathbf{b}}'_1, \widehat{\mathbf{b}}'_2)$. This angle depends on space position, in which the static basis is defined. The following coordinate transformation comes out from the change of the static basis origin: $\mathbf{x} \to \mathbf{x}'$, which changes the gyrophase angle definition $\theta = \theta' + \xi(\mathbf{x}')$. It is evident that the dynamics should be independent from the gyro-gauge angle, because the dynamics is intrinsic with respect to the choice of the basis vectors. The gyro-gauge vector, associated with derivation of static basis $\mathbf{R} = \boldsymbol{\nabla}\widehat{\mathbf{b}}_1 \cdot \widehat{\mathbf{b}}_2$ may appear into the phase space Lagrangian expression, undergoing the following transformation with the coordinate change: $\mathbf{R}' = \boldsymbol{\nabla}'\widehat{\mathbf{b}}'_1 \cdot \widehat{\mathbf{b}}'_2 = \mathbf{R} + \boldsymbol{\nabla}\xi$, which breaks basis invariance of dynamics. However we remark that some operations involving the gyro-gauge vector will not lead to the breaking of that invariance:

$$\boldsymbol{\nabla} \times \mathbf{R}' = \boldsymbol{\nabla} \times (\mathbf{R} + \boldsymbol{\nabla}\xi) = \boldsymbol{\nabla} \times \mathbf{R}, \tag{6}$$

therefore such a quantity is allowed to enter the dynamical description. In early works of Littlejohn the gyrophase dependence invariance has not been taken into the account, that issue is corrected in further works. We remark as well, that the norm of the gyrophase vector $\mathbf{R}$ is of the order of ϵ_B.

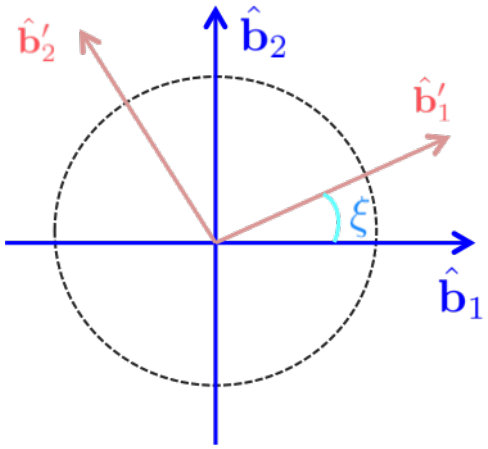

FIG. 2: Gyrophase angle ξ dependence of stationary basis vectors.

C. Dynamical reduction, first step: guiding-center dynamics

We are now proceeding with detailed description of the dynamical reduction procedure. The first step considers the effects of the background magnetic field only on the guiding-center dynamics.

The guiding-center dynamical reduction is time-independent. This is why it is performed on 6 dimensional phase space, consisting of the parallel kinetic momentum $p_\parallel = mv_\parallel$, reduced (guiding-center) position $\mathbf{X}$ (the effective radius of particle rotation above the magnetic field line), magnetic moment μ and the gyroangle θ. The corresponding particle's Lagrangian writes as follows:

$$\Lambda_{\rm gc} = \frac{c}{e}\mathbf{A}^* \cdot \dot{\mathbf{X}} + \mu\, \dot{\theta} - H_{\rm gc}, \tag{7}$$

where the symplectic part contains modified magnetic potential. In that sense, one can say that the guiding-center change of coordinates generates mixing between space and kinetic parts of the phase space.

$$\mathbf{A}^* = \mathbf{A} + \frac{c}{e}\left(p_\parallel \widehat{\mathbf{b}} - m\mu\mathbf{R}\right), \tag{8}$$

with $\widehat{\mathbf{b}}$ the normalized direction of background magnetic field $\mathbf{B}$; the gyro-gauge vector $\mathbf{R}$ and the

guiding-center Hamiltonian is given by:

$$H_{\rm gc} = \frac{p_\parallel^2}{2m} + \mu B, \tag{9}$$

where B is the modulus of the background magnetic field.

We remark that the gyro-gauge vector $\mathbf{R}$ appears into the definition of the magnetic vector $\mathbf{A}^*$, which is defined up to the gauge transformation in any cases. This dependence is fixed with definition of the modified magnetic field $\mathbf{B}^* = \boldsymbol{\nabla} \times \mathbf{A}^*$, which is invariant under the gyro-gauge transformation accordingly to (6).

By inverting the symplectic matrix, which corresponds to particle's Lagrangian (7) we obtain the following guiding-center Poisson bracket:

$$\{F, G\}_{\rm gc} = \frac{e}{mc}\left(\frac{\partial F}{\partial \theta}\frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu}\frac{\partial G}{\partial \theta}\right) + \frac{\mathbf{B}^*}{B_\parallel^*}\cdot\left(\boldsymbol{\nabla} F \frac{\partial G}{\partial p_\parallel} - \frac{\partial F}{\partial p_\parallel}\boldsymbol{\nabla} G\right) - \frac{c\widehat{\mathbf{b}}}{eB_\parallel^*}\cdot(\boldsymbol{\nabla} F \times \boldsymbol{\nabla} G), \tag{10}$$

Remark that the condition $\boldsymbol{\nabla}\cdot\mathbf{B}^* = 0$ guarantees the Liouville theorem (i.e. conservation of the phase space volume) on the reduced phase space, see for details [5].

The characteristics of the guiding-center dynamics are derived from the Hamiltonian equations as follows:

$$\dot{\mathbf{X}} = \{\mathbf{X}, H_{\rm gc}\}_{\rm gc} = \frac{p_\parallel}{m}\frac{\mathbf{B}^*}{B_\parallel^*} + \frac{c\widehat{\mathbf{b}}}{eB_\parallel^*}\times\mu\nabla B \tag{11}$$

$$\dot{p_\parallel} = \{p_\parallel, H_{\rm gc}\}_{\rm gc} = -\mu\nabla B\cdot\frac{\mathbf{B}^*}{B_\parallel^*} \tag{12}$$

$$\dot{\theta} = \frac{eB}{mc} \tag{13}$$

$$\dot{\mu} = 0. \tag{14}$$

One can immediately conclude that the fastest scale of motion (13) is totally uncoupled from the reduced position (11) and kinetic momentum dynamics (12). In the same time, the magnetic moment μ has trivial dynamics on the reduced phase space.

D. Dynamical reduction, second step: gyrocenter dynamics

In the framework of two-step gyrokinetic reduction procedure, external time-dependent electromagnetic fields are introduced into the system at the second step of dynamical reduction: the *gyrocenter* step.

In this case, time becomes a dynamical variable of the system. Therefore, one needs to expand the guiding-center phase space by introducing a couple of canonically conjugated variables (w, t).

The latter leads to natural extension of particle's phase space up to 8 dimensions. On the extended phase space, the extended non-perturbed Hamiltonian is $\mathcal{H}_{\rm gc} \equiv H_{\rm gc} - w$ and the Poisson bracket (10) has an additional canonical term:

$$\begin{aligned}\{F,G\}_{\rm ext} \;=\; & \frac{e}{mc}\left(\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\theta}\right) + \frac{\mathbf{B}^*}{B^*_\parallel}\cdot\left(\boldsymbol{\nabla}F\frac{\partial G}{\partial p_\parallel} - \frac{\partial F}{\partial p_\parallel}\boldsymbol{\nabla}G\right) \\ & - \frac{c\widehat{\mathbf{b}}}{eB^*_\parallel}\cdot(\boldsymbol{\nabla}F\times\boldsymbol{\nabla}G) - \frac{\partial F}{\partial w}\frac{\partial G}{\partial t} + \frac{\partial F}{\partial t}\frac{\partial G}{\partial w}. \end{aligned} \tag{15}$$

In this work, we choose to keep the latter expression for the Poisson bracket free from introducing any fluctuating fields. This is the common choice, which has been already adopted while derivation of the model for the code ORB5. Therefore, all the effects from the dynamical reduction must be accounted inside the expression for the reduced gyrocenter Hamiltonian $H_{\rm gy}$.

The expression for $H_{\rm gy}$, which we are using further on for constructing coupled model with electromagnetic fields is derived in the following section up to the second order in ϵ_δ.

Before we proceed with that derivation, here we discuss the relationship between using new reduced particle position (the gyrocenter position), obtained via the reduction procedure and the polarization corrections, it induces into the reduced gyrocenter Hamiltonian $H_{\rm gy}$.

1. Polarization effects: relationship between coordinate transformation and reduced Hamiltonian dynamics

As we have already mentioned above, the systematic reduction procedure applied to the particle phase space Lagrangian provides us a set of new coordinates, in which we describe the reduced particle dynamics. Following that general reduction procedure which involves the Lie derivatives techniques [5], we discover that the definition of new coordinates and identification of polarization corrections due to the dynamical reduction into the reduced Hamiltonian are intimately related to each other. We do not aim to expose the general Lie-transforms reduction procedure here, it can be found in compact form in [5], detailed derivation or the guiding- center part of the reduction can be found in [17].

Here we adopt the general attitude for code models derivation: we push the curvature effects of magnetic field at the next order with respect to the amplitude of the electromagnetic fluctuations: $\epsilon_B = \epsilon\,\epsilon_\delta$, where $\epsilon < 1$ is a free small parameter, which can also depend on ϵ_δ. Of course, from the point of view of transition to the limit different choices of ϵ will lead to a different model from the general point of view. Here we will drop our series decompositions at the second order in ϵ_δ and the first order in ϵ_B, which lead to the same result while $\epsilon_B = \epsilon_\delta^2$ or $\epsilon_B = \epsilon\,\epsilon_\delta$. Remark that the norm of the $\boldsymbol{\rho}_0$ is $\mathcal{O}(\epsilon_B^0)$ and those of $\boldsymbol{\rho}_0\cdot\boldsymbol{\nabla}\boldsymbol{\rho}_0$ is $\mathcal{O}(\epsilon_B)$.

In this subsection we are concentrating on the explicit derivation of the reduced Hamiltonian, by assuming that we have performed the dynamical reduction at the lowest order of the guiding-center and the gyrocenter transformations.

In the other words, it means that the transition between the initial non-reduced particle position $\mathbf{x}$ and the reduced particle position $\mathbf{X}$ is defined as follows:

$$\mathbf{x} \equiv \mathbf{X} + \boldsymbol{\rho}_0(\mathbf{X}, \mu, \theta) + \boldsymbol{\rho}_1(\mathbf{X}, \mu, \theta), \tag{16}$$

with $\boldsymbol{\rho}_0$ corresponding to the lowest order guiding-center displacement and $\boldsymbol{\rho}_1$ the lowest order of the gyrocenter displacement. Let us now define each of them explicitly.

The lowest order guiding-center displacement is given by:

$$\boldsymbol{\rho}_0 \equiv \frac{mc}{e}\sqrt{\frac{2\mu}{mB}}\,\widehat{\boldsymbol{\rho}} \equiv \rho_0\widehat{\boldsymbol{\rho}}. \tag{17}$$

This displacement takes into account the background magnetic field $\mathbf{B}$, which is locally uniform. We emphasize that the amplitude of background magnetic field B is evaluated at the reduced guiding-center position $\mathbf{X}$ and $\widehat{\boldsymbol{\rho}}$, is one of the dynamical basis vectors, defined in (3). All the following corrections to the guiding-center displacement are related to the magnetic curvature.

The expression for the first order gyrocenter displacement is given by:

$$\boldsymbol{\rho}_1 \equiv -2\frac{e}{m\Omega^2}\left(\widehat{\boldsymbol{\rho}}\widehat{\boldsymbol{\rho}}\cdot\boldsymbol{\nabla}\psi_1(\mathbf{X})\right), \tag{18}$$

here $\psi_1 = \phi_1 - \frac{e}{mc}\,p_z A_{1\|}$ represents linear electromagnetic potential, it means that $\boldsymbol{\rho}_1$ contains both gyroaveraged and fluctuating parts.

Deriving the reduced expression for the Hamiltonian is the key point in derivation of the reduced Maxwell-Vlasov model. Namely, it will allow us to define the polarization effects (i.e. effects due to the dynamical reduction) into the reduced Maxwell equations later on from the variational principle.

We will show that for identification of the second order corrections in ϵ_δ to the Hamiltonian requires coordinate transformation of the first order in ϵ_δ.

First, we proceed with explicit derivation of the first order gyrocenter displacement $\boldsymbol{\rho}_1$ in the following section.

2. Second order reduced Hamiltonian

We will show the effects of two coordinate transformations (16) into the reduced Hamiltonian.

We start with constructing the perturbed electromagnetic Hamiltonian $H_{\rm gy}$ of the second order in ϵ_δ.

We perform the following transformation of the guiding-center Hamiltonian $H_{\rm gc}$ (9): first, we introduce the definition of new perturbed canonical momentum coordinate:

$$p_z = p_\parallel + \frac{e}{c} A_{1\parallel}(\mathbf{X} + \underbrace{\boldsymbol{\rho}_0 + \boldsymbol{\rho}_1}_{\equiv \boldsymbol{\rho}_\epsilon}), \tag{19}$$

second, we also add a perturbed electrostatic potential $\phi_1(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1)$. Remark that this change of coordinates will not affect the definition of the Poisson bracket (15) because $\partial_{p_z} \equiv \partial_{p_\parallel}$. In the other words, we decide here to add the perturbed part of the magnetic potential $A_{1\parallel}$ into the Hamiltonian and not into the Poisson bracket. This is the Hamiltonian representation of the gyrocenter dynamics. We discuss other possible representations of the gyrocenter dynamics below.

The perturbed gyrocenter Hamiltonian is:

$$H_{\rm gy} = \frac{p_z^2}{2m} + \mu B + \left(e\ \phi_1(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1) - \frac{1}{m}\ \frac{e}{c}\ p_z\ A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1) \right) + \frac{1}{2m} \left(\frac{e}{c}\right)^2 A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1)^2, \tag{20}$$

where the linear perturbed gyrocenter potential :

$$\psi_{\rm 1gy} \equiv\ \phi_1(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1) - \frac{1}{mc}\ p_z\ A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1). \tag{21}$$

Here, $\boldsymbol{\rho}_0$ represents the lowest order guiding-center contribution and $\boldsymbol{\rho}_1$ the gyrocenter contribution correspondingly. We are performing expansion of $\boldsymbol{\rho}_1$ corrections:

$$\psi_{\rm 1gy} \equiv \psi_1(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1)\ =\ \psi_1\left(\mathbf{X} + \boldsymbol{\rho}_0\right) + \boldsymbol{\rho}_1(\mathbf{X} + \boldsymbol{\rho}_0) \cdot \boldsymbol{\nabla}\psi_1\left(\mathbf{X} + \boldsymbol{\rho}_0\right) + \mathcal{O}(\epsilon_\delta^2) \tag{22}$$

$$\equiv\ \psi_{\rm 1gc} + \boldsymbol{\rho}_1 \cdot \boldsymbol{\nabla}\psi_{\rm 1gc} + \mathcal{O}(\epsilon_\delta^2) \tag{23}$$

In what concerns the purely electromagnetic quadratic contribution $A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1)^2$, here only corrections related to the guiding-center transformation are taken into the account, because all the polarization effects related to $\boldsymbol{\rho}_1$ are at least of the order $\mathcal{O}(\epsilon_\delta^3)$, i.e. $A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1)^2 \simeq A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0)^2$.

The gyro-averaged gyrocenter Hamiltonian, corresponding to the change of coordinates from the full particle position $\mathbf{x}$ to the reduced position $\mathbf{X}$ with taking into account highest order displacements $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_1$ is given by:

$$\langle H_{\rm gy} \rangle\ =\ \frac{p_z^2}{2m} + \mu B + \epsilon_\delta \left\langle \psi_{\rm 1gc} \right\rangle + \epsilon_\delta^2 \left\langle \boldsymbol{\rho}_1 \cdot \boldsymbol{\nabla}\psi_{\rm 1gc} \right\rangle + \frac{\epsilon_\delta^2}{2m} \left(\frac{e}{c}\right)^2 \left\langle A_{1\parallel{\rm gc}}^2 \right\rangle \tag{24}$$

Here the gyro averaging operations are performed with respect to the lowest order guiding-center displacement $\boldsymbol{\rho}_0$. It means that the gyro averaging is performed on uniform circles of radius ρ_0:

$$\langle \psi_{1\mathrm{gc}} \rangle = \langle \psi_1 \left(\mathbf{X} + \boldsymbol{\rho}_0(\mathbf{X}, \mu, \theta) \right\rangle \equiv \int_0^{2\pi} d\theta \; \psi_1 \left(\mathbf{X} + \boldsymbol{\rho}_0(\mathbf{X}, \mu, \theta) \right), \tag{25}$$

Let us now explicitly evaluate the expression, which contains only highest order guiding-center FLR contributions to the mixed electromagnetic term $\langle \boldsymbol{\rho}_1 \cdot \boldsymbol{\nabla} \psi_{1\mathrm{gc}} \rangle$. Such a derivation will lead us to the model, which is currently implemented into the ORB5 code. Such a model is valid within the *long wave lenght* approximation, i.e. for the case with $k_\perp \rho \ll 1$.

The gyro averaging of the full expression for the gyrocenter potential gives :

$$\langle \psi_{1\mathrm{gy}} \rangle \; = \; \langle \psi_{1\mathrm{gc}} \rangle + \langle \boldsymbol{\rho}_1 \cdot \boldsymbol{\nabla} \psi_{1\mathrm{gc}} \rangle + \ldots, \tag{26}$$

in case, when only the lowest order contribution to $\psi_{1\mathrm{gc}}$, is taken into the account: $\langle \boldsymbol{\rho}_1(\mathbf{X} + \boldsymbol{\rho}_0) \cdot \boldsymbol{\nabla} \psi_1(\mathbf{X}) \rangle = \langle \boldsymbol{\rho}_1(\mathbf{X} + \boldsymbol{\rho}_0) \rangle \cdot \boldsymbol{\nabla} \psi_1(\mathbf{X})$.

The gyro averaged part of the gyrocenter displacement (18) is calculated with using that

$$\langle \widehat{\boldsymbol{\rho}} \widehat{\boldsymbol{\rho}} \rangle = \frac{1}{2} \; \mathbf{1}_\perp, \tag{27}$$

where $\mathbf{1}_\perp = \widehat{\perp}\widehat{\perp} + \widehat{\boldsymbol{\rho}}\widehat{\boldsymbol{\rho}} \equiv \mathbf{1} - \widehat{\mathbf{b}}\widehat{\mathbf{b}}$ is the perpendicular dyadic tensor. Therefore,

$$\langle \boldsymbol{\rho}_1 \rangle = - \left(\frac{e}{m\Omega^2} \right) \boldsymbol{\nabla}_\perp \psi_1(\mathbf{X}). \tag{28}$$

The following expression accounts only highest order contributions to the nonlinear mixed electromagnetic term $\langle \boldsymbol{\rho}_1 \cdot \boldsymbol{\nabla} \psi_1 \rangle$. This is the second order Hamiltonian, which will be later compared to the one implemented into the ORB5 code.

$$\begin{aligned} \langle H_{\mathrm{gy}} \rangle \; &= \; \frac{p_z^2}{2m} + \mu B + \epsilon_\delta \left(e \left\langle \phi_1(\mathbf{X} + \boldsymbol{\rho}_0) \right\rangle - \frac{1}{m} \frac{e}{c} \; p_z \; \left\langle A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0) \right\rangle \right) \\ &+ \; \frac{\epsilon_\delta^2}{2m} \left(\frac{e}{c} \right)^2 \left\langle A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0)^2 \right\rangle - \frac{\epsilon_\delta^2}{2} \; \frac{e}{m\Omega^2} \left| \boldsymbol{\nabla}_\perp \phi_1(\mathbf{X}) - \frac{e}{c} \; p_z \boldsymbol{\nabla}_\perp A_{1\parallel}(\mathbf{X}) \right|^2. \end{aligned} \tag{29}$$

This expression for the gyrocenter Hamiltonian contains gyrocenter corrections of orders $\mathcal{O}(\epsilon_\delta)$ and $\mathcal{O}(\epsilon_\delta^2)$. The gyro averaging here is realized with respect to the lowest order guiding-center displacement $\boldsymbol{\rho}_0$ accordingly to (25) . It means that the gyro averaged quantities contain the FLR corrections related to the lowest order guiding-center displacement $\boldsymbol{\rho}_0$ at all orders. It takes into account the *long wave lenght* approximation for the last nonlinear electromagnetic term.

The first order term represents linear gyro averaged electromagnetic potential $\langle \psi_{1\mathrm{gy}} \rangle$. The second order term has two parts: the purely magnetic one $\left\langle A_{1\parallel}^2 \right\rangle$ and the electromagnetic contribution evaluated into the gyrocenter position $\mathbf{X}$, which does not contain any other FLR corrections with respect to the guiding-center displacement $\boldsymbol{\rho}_0$.

3. Symplectic and Hamiltonian representations of gyrocenter reduction

The electrostatic part of fields perturbation ϕ_1 is usually accounted as a part of a perturbed gyrocenter Hamiltonian. However, there exist several possibilities for organizing the dynamical reduction procedure depending on where the magnetic potential perturbation $A_{1\|}$ is taken into account: into the Symplectic or into the Hamiltonian part of the phase space Lagrangian (1).

The Hamiltonian representation includes parallel magnetic perturbation $A_{1\|}$ inside the expression for the perturbed Hamiltonian and leaves the guiding-center Poisson bracket (15) invariant. As we have seen into the previous section, it uses parallel canonical gyrocenter momentum p_z as one of the phase space variables, it is therefore sometimes called "p_z representation".

The Symplectic representation accounts the perturbed parallel magnetic moment inside the Symplectic part of the phase-space Lagrangian, which leads to perturbation of guiding-center Poisson bracket (15). In that case, parallel kinetic momentum is unperturbed $p_\| = mv_\|$. The latter facilitates identification of various physical terms and makes possible to avoid the cancellation problem [9], related to presence of terms with several orders of difference inside the corresponding Ampère's equation.

In what concerns PIC codes, Hamiltonian representation is preferable, since in the symplectic one the inductive electric field (i.e. explicit time-derivative of perturbative magnetic potential) appears into the characteristics. It therefore requiers additional implicit time integrator. When the Hamiltonian representation is chosen, the explicit time derivatives of perturbed potentials are only contained into the dynamics of w-variable and therefore completely uncoupled from the dynamics of the physical reduced phase space. Symplectic representation is used for derivation of particles characteristics and Vlasov equation implemented in Eulerian , i.e. GENE code [10], [8].

a. First order gyrocenter characteristics First order particle characteristics in p_z representation are derived from the first order in ϵ_δ gyrocenter Hamiltonian:

$$\mathcal{H}^{(1)}_{\rm gy} = H_{\rm gc} + e\ \epsilon_\delta \left\langle \phi_{1\rm gc} \right\rangle - \frac{\epsilon_\delta}{m}\frac{e}{c}\ p_z \left\langle A_{1\|gc} \right\rangle - w, \tag{30}$$

and the non-perturbed guiding-center Poisson bracket on extended 8-dimensional phase space (15):

$$\dot{\mathbf{X}}^{(1)}_{\rm gy} \ = \ \left\{ \mathbf{X}, \mathcal{H}^{(1)}_{\rm gy} \right\}_{\rm ext} = \frac{\mathbf{B}^*}{B^*_\|}\frac{\partial \mathcal{H}^{(1)}_{\rm gy}}{\partial p_z} + \frac{c\hat{\mathbf{b}}}{eB^*_\|} \times \nabla \mathcal{H}^{(1)}_{\rm gy} \tag{31}$$

$$\dot{p}^{(1)}_{\rm z\ gy} \ = \ \left\{ p_z, \mathcal{H}^{(1)}_{\rm gy} \right\}_{\rm ext} = -\frac{\mathbf{B}^*}{B^*_\|} \cdot \nabla \mathcal{H}^{(1)}_{\rm gy}. \tag{32}$$

We give detailed derivation of characteristics equations in Hamiltonian representation as it is implemented in ORB5 code in the Appendix B; together with ORB5 code diagnostics.

In Symplectic representation only the electrostatic part of the perturbation is included inside the expression for the gyrocenter perturbed Hamiltonian $\mathcal{H}^s_{\rm gy}$, while the magnetic part of the perturbation is accounted into the symplectic part of the Lagrangian (1) through the symplectic potential (8) and therefore into the Poisson bracket, which we name here with indice gy. The latter makes appear explicit time derivatives of magnetic potential into the expressions for reduced phase space characteristics $(\mathbf{X}, p_\parallel)$:

$$\dot{\mathbf{X}}^{(1,s)}_{\rm gy} = \left\{\mathbf{X}, \mathcal{H}^s_{\rm gy}\right\}_{\rm gy} = \frac{\mathbf{B}^*}{B^*_\parallel}\frac{\partial \mathcal{H}^s_{\rm gy}}{\partial p_\parallel} + \frac{c\hat{\mathbf{b}}}{eB^*_\parallel} \times \nabla \mathcal{H}^s_{\rm gy} \tag{33}$$

$$\dot{p_\parallel}^{(1,s)}_{\rm gy} = \left\{p_\parallel, \mathcal{H}^s_{\rm gy}\right\}_{\rm gy} = -\frac{\mathbf{B}^*}{B^*_\parallel} \cdot \nabla \mathcal{H}^s_{\rm gy} - \frac{e}{c}\frac{\partial \left\langle A_{1\parallel gc}\right\rangle}{\partial t}$$

We also remark that the symplectic magnetic field $\mathbf{B}^*$ now contains a part of perturbed magnetic field and therefore it enters into the Liouville condition for the phase space volume conservation $\boldsymbol{\nabla} \cdot \mathbf{B}^* = 0$.

4. Field-particles coupling on the reduced phase space

Gyrokinetic coordinate transformation introduces a difference between positions of fields and particles: fields are evaluated into the non-reduced spatial position $\mathbf{r}$, while particles are now replaced by gyrocentres $\mathbf{X}$, whose position differs on the distance of the polarization displacement $\boldsymbol{\rho}_\epsilon \equiv \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1$. In order to correctly account field-particles interaction on the reduced phase space, one should include contributions from all gyrocenters $\mathbf{X}$ located at the distance $\boldsymbol{\rho}_\epsilon$ from the field position $\mathbf{r}$. On the figure 6 we represent this statement graphically. As we could see in the previous section, all effects from the lowest order gyrocenter displacement $\boldsymbol{\rho}_1$ are explicitly accounted into the reduced Hamiltonian $H_{\rm gy}$ and all the averaging operators contain contributions only from the guiding-center displacement $\boldsymbol{\rho}_0$. Therefore, the polarization effects associated with the gyrocenter displacement will appear into the reduced characteristics, while the polarization effects coming from the guiding-center displacement are included into the averaged electromagnetic potentials $\left(\left\langle \phi_{1\rm gc}\right\rangle, \left\langle A_{1\parallel gc}\right\rangle\right)$. To compactify our formulas in further calculations we use the following notation to account the guiding-center polarization effects:

$$\delta^3_{\rm gc} \equiv \delta^3\left(\mathbf{X} + \boldsymbol{\rho}_0(\mathbf{X}, \mu, \theta) - \mathbf{r}\right). \tag{34}$$

We remark that this operator is not symmetric in general with respect to the transitions between the reduced and non-reduced coordinates: i.e. $\boldsymbol{\rho}_0$ can be seen as a function of $\mathbf{X}$ or $\mathbf{x}$.

A point that should be clarified here is about the transitions between the reduced guiding-center coordinate and the original particle's position. At the lowest order of transformation, we have:

$$\boldsymbol{\rho}_0((\mathbf{x}-\mathbf{X})+\mathbf{X}) \sim \boldsymbol{\rho}_0(\mathbf{X}) + (\mathbf{x}-\mathbf{X})\cdot\boldsymbol{\nabla}\boldsymbol{\rho}_0 + \dots$$

The distance between the gyrocenter position $\mathbf{X}$ and the non-reduced particle position $\mathbf{x}$ is supposed to be small (let name it ϵ), after all we just perform near-identinty transformations. In the same time, the second rank tensor $\boldsymbol{\nabla}\boldsymbol{\rho}_0$ contains $\mathcal{O}(\epsilon_B)$ contributions. Then, the first correction in that series decomposition is at least of order $\mathcal{O}(\epsilon_B\epsilon)$ and therefore can be therefore neglected in our further calculations. Finally, we see that at the lowest order of coordinate transformation, the δ^3_{gc} operator is symmetric. This will be important for further calculation of functional derivatives.

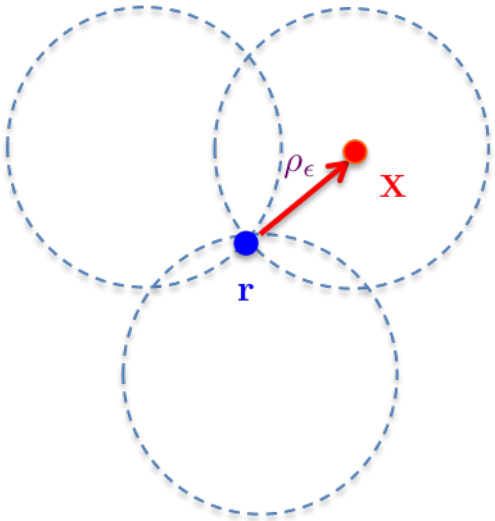

FIG. 3: Gyrokinetic polarization: contributions of reduced particles positions $\mathbf{X}$, situated at the distance of polarization displacement $\boldsymbol{\rho}_\epsilon$ from the field position $\mathbf{r}$.

III. EULERIAN SECOND ORDER VARIATIONAL PRINCIPLE: GENERAL METHOD

As it has been already mentioned in the introduction, there exist two types of variational formulation for plasma dynamics. The main difference consists in treatment of particles dynamics. While the Eulerian formulation allows to consider Vlasov field as a dynamical field, the Lagrangian formulation treats particles dynamics through their characteristics.

In this section we introduce an implicit expression of the second order Eulerian action functional, i.e. containing corrections up to the $\mathcal{O}(\epsilon_\delta^2)$ and up to the second order in guiding-center FLR effects related to the lowest order displacement $\boldsymbol{\rho}_0$. The implicit expression of this Eulerian action functional has been obtained in [4] via systematic truncation of full Eulerian gyrokinetic action [3].

The second order Maxwell-Vlasov action functional writes:

$$\begin{aligned}\mathcal{I}^{\mathcal{E}}[\phi_1, \mathbf{A}_1, \mathcal{F}] \;\equiv\; \int_{t_1}^{t_2} \mathcal{A}^{\mathcal{E}}[\phi_1, \mathbf{A}_1, \mathcal{F}]\; dt = \int \frac{dV\; dt}{8\pi} \left(\epsilon_\delta^2 \left|\mathbf{E}_1\right|^2 - \left|\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1\right|^2\right) \\ - \int\; d^8Z\; \mathcal{F}\; \mathcal{H}_1 - \int\; d^6Z\; dt\; F_0\; H_2\end{aligned} \tag{35}$$

The first term here represents the Maxwellian part of the action with electrostatic field $\mathbf{E}_1 \equiv -\boldsymbol{\nabla}\phi_1 - c^{-1}\partial_t \mathbf{A}_1$. The magnetic field is separated into background $\mathbf{B}_0$ and fluctuating (dynamical) part $\mathbf{B}_1 \equiv \boldsymbol{\nabla} \times \mathbf{A}_1$.

The second and third terms are contributions to the Vlasov part of the action. The first Vlasov term contains dynamical part of the distribution function and it is defined on the extended 8-dimensional phase space with $d^8Z \equiv\; B_\parallel^* dV\; dp_z\; d\mu\; d\theta\; dt\; dw$,where dV represents space volume element; while the second Vlasov term is defined on the n-dimensional reduced gyrocenter phase space with $d^6Z \equiv\; B_\parallel^* dV\; dp_z\; d\mu\; d\theta$. This manipulation is necessary to get the Vlasov equation from the variational principle. This is also mandatory for derivation of the conservation laws via the Noether's method.

The second term of the action functional contains the truncated first order Vlasov distribution on the extended phase space:

$$\mathcal{F} \equiv \left(F_0 + \epsilon_\delta\; F_1\right) \delta\left(w - H_0 - \epsilon_\delta H_1\right), \tag{36}$$

while the third term keeps only a non-dynamical part of the Vlasov distribution F_0. Therefore, in our action functional the second order reduced gyrocenter dynamics, generated by the Hamiltonian H_2 is associated with background distribution function only.

In this work we aim to compare results of our derivation (i.e. equations of motion and conservation laws) for the Eulerian action functional (35) with the action functional used for construction of the ORB5 code model. We describe that model in section (V). Finally, we aim to build up an exactly conserved electromagnetic energy invariant via the Noether's method. We discuss that issue in the section (III C). In that section we proceed with implicit derivation without giving expressions for H_1 and H_2. In the next section we explicit the expressions for the reduced particle dynamics model and provide explicit expressions for the corresponding equations of motion (Gyrokinetic Maxwell-Vlasov system) and the conserved energy density.

A. First variation of the action functional

In this section we give the procedure of the first variation calculation for the second order Eulerian action (35). The variation calculation requires the use of the functionals derivatives.

For a functional $\mathcal{L} = \int d\Lambda \ \mathsf{L}\,(\eta, \nabla\eta)$ depending on a scalar field $\eta = \eta(\mathbf{x})$ and its gradient $\nabla\eta = \nabla\eta(\mathbf{x})$, the functional derivative is :

$$\frac{\delta\mathcal{L}}{\delta\eta} \circ \chi \equiv \frac{d}{d\nu}\bigg|_{\nu=0} \left(\int d\Lambda \ \mathsf{L}\,[\eta + \nu\chi, \nabla\eta + \nu\nabla\chi] \right) = \int d\Lambda \ \frac{\partial \mathsf{L}}{\partial \eta} \circ \chi + \int d\Lambda \ \frac{\partial \mathsf{L}}{\partial \nabla\eta} \circ \nabla\chi \tag{37}$$

$$= \int d\Lambda \ \left(\frac{\partial \mathsf{L}}{\partial \eta} - \left(\nabla \cdot \frac{\partial \mathsf{L}}{\partial \nabla\eta} \right) \right) \circ \chi + \int d\Lambda \ \nabla \cdot \left(\frac{\partial \mathsf{L}}{\partial \nabla\eta} \circ \chi \right), \tag{38}$$

where χ is an arbitrary test function.

Therefore, the first variation for the second order Eulerian action is:

$$\delta\mathcal{I}^{\mathcal{E}} \equiv \int_{t_1}^{t_2} \delta\mathcal{A}^{\mathcal{E}}\,[\phi_1, \mathbf{A}_1, \mathcal{F}]\,dt = \int_{t_1}^{t_2} \left(\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\phi_1} \circ \widehat{\phi}_1 + \frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\mathbf{A}_1} \circ \widehat{\mathbf{A}}_1 + \frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\mathcal{F}} \circ \delta\widehat{\mathcal{F}} \right) dt, \tag{39}$$

Such a derivation provides two important informations: first of all, it allows one to get a system of coupled Maxwell-Vlasov equations; second, it also provides us the expressions for the Noether's terms, necessary for the corresponding conservation laws derivation. The Noether's terms are represented by exact derivatives and do not contribute to the dynamical part.

To that purpose, while evaluating the first variation of the action functional $\mathcal{I}^{\mathcal{E}}[\phi_1, \mathbf{A}_1, \mathcal{F}]$, we are using the following definition of the functional derivative which makes explicitly appear dependencies on the scalar fields and their gradients.

As we will see, the equation (37) defines dynamics in a *weak* form, while the equation (38) obtained via the Leibnitz rule application contains the *dynamical term* (multiplied by χ) and the *Noether's term*, which we are using for derivation of the conservation laws.

We remark here that from the point of view of mathematical definition $\delta\mathcal{L}/\delta\eta$ is a linear functional, i.e. in order to get a numerical value, one has to apply it to the test function (we named it χ above), which does not have a small norm a priori. In physics, this test function is traditionally named with the same letter as the derivative, see for example [16],[5], so in our case it would be $\delta\eta$. This is used in order to keep an intuitional link to the definition of Riemann's derivative. We will be using such a definition while deriving the equations of motion for the Eulerian action (35).

We also remark here that the spatial derivative ∇ can be also replaced by the time derivative ∂_t into the expressions (37) and (38) respectively. We will be using this fact for energy conservation law derivation in the following section.

1. Constrained Eulerian variations

Before proceeding with explicit first variation derivation for Eulerian action functional (35), we need to explain how Vlasov field variations are calculated. Indeed, variation of the extended phase space Vlasov field $\mathcal{F}$ is evaluated in the following constrained form:

$$\delta\mathcal{F} \equiv \{\mathcal{S}, \mathcal{F}\}_{\text{ext}}, \tag{40}$$

where $\mathcal{S}$ is a generating function, not affecting the equations of motion but defining the conservation law for the reduced system through the Noether's theorem. In what concerns functional derivatives with respect to the electrostatic and the electromagnetic potential, their evaluation is performed with respect to the general rule of classical field theory. The electric and magnetic field variations should respect the constraints imposed by electromagnetic fields definitions :

$$\delta\mathbf{E}_1 \;=\; -\boldsymbol{\nabla}\delta\phi_1 - c^{-1}\partial_t\delta\mathbf{A}_1 \tag{41}$$

$$\delta\mathbf{B}_1 \;=\; \boldsymbol{\nabla}\times(\delta\mathbf{A}_1)\,. \tag{42}$$

For convenience of further calculations we separate the general second order action functional on its field part and Vlasov (Vl) part :

$$\mathcal{A}^{\mathcal{E}}[\phi_1, \mathbf{A}_1, \mathcal{F}] \equiv \mathcal{A}^{\mathcal{E},(\mathsf{field})}[\phi_1, \mathbf{A}_1] + \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}[\phi_1, \mathbf{A}_1, \mathcal{F}] = \tag{43}$$
$$\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}[\phi_1, \mathbf{A}_1] + \mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{magn}}[\phi_1, \mathbf{A}_1] + \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}[\phi_1, \mathbf{A}_1, \mathcal{F}] + \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}}[\phi_1, \mathbf{A}_1]$$

The field part of the action is further divided into the electric Maxwellian part:

$$\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}[\phi_1, \mathbf{A}_1] \equiv \frac{\epsilon_\delta^2}{8\pi}\int dV\;\; |\mathbf{E}_1|^2 \tag{44}$$

and magnetic Maxwellian part:

$$\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{magn}}[\phi_1, \mathbf{A}_1] \equiv -\frac{1}{8\pi}\int dV\;\; |\mathbf{B}_0 + \epsilon_\delta\mathbf{B}_1|^2 \tag{45}$$

We separate the linear and nonlinear contributions to the Vlasov part of the action functional in the following way. The linear one: contains contribution from the dynamical part of the extended Vlasov field $\mathcal{F}$ defined by (36) and the first order gyrocenter Hamiltonian $\mathcal{H}_1$ given by (30).

$$\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}[\phi_1, \mathbf{A}_1, \mathcal{F}] \equiv -\int\;\; d^8Z\;\mathcal{F}\;\mathcal{H}_1 \tag{46}$$

Finally, the nonlinear Vlasov term contains only non-dynamical part of the Vlasov field F_0 and the second order extended electromagnetic Hamiltonian H_2, which will be defined below accordingly to the different Maxwell-Vlasov gyrokinetic models we are about to derive:

$$\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}}\left[\phi_1,\mathbf{A}_1\right] \equiv -\int\, d^6Z\; dt\; F_0\; H_2 \tag{47}$$

2. *Fields contributions*

We start with calculating functional derivative of fields contribution to the action functional.

$$\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})} \equiv \frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}}{\delta\phi_1}\circ\widehat{\phi}_1 + \frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1, \tag{48}$$

where $\widehat{\phi}_1$ and $\widehat{\mathbf{A}}_1$ are the test functions.

First, for the electrostatic field term with $\mathbf{E}_1$ we have:

$$\begin{aligned}
\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}}{\delta\phi_1}\circ\widehat{\phi}_1 &= \frac{d}{d\nu}\left[\int\frac{dV}{8\pi}\;\epsilon_\delta^2\left|\mathbf{E}_1-\nu\boldsymbol{\nabla}\widehat{\phi}\right|^2\right]|_{\nu=0}\\
&= \underbrace{-\epsilon_\delta^2\int\frac{dV}{4\pi}\;\mathbf{E}_1\cdot\boldsymbol{\nabla}\widehat{\phi}_1}_{\mathsf{Weak\ dynamics}}\\
&= \underbrace{-\epsilon_\delta^2\int\frac{dV}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E}_1\;\widehat{\phi}_1}_{\mathsf{Dynamical\ term}} + \underbrace{\epsilon_\delta^2\int\frac{dV}{4\pi}\boldsymbol{\nabla}\cdot\left(\mathbf{E}_1\;\widehat{\phi}_1\right)}_{\mathsf{Noether's\ term}},
\end{aligned} \tag{49}$$

an electrostatic part of functional derivative and

$$\begin{aligned}
\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1 = \frac{d}{d\nu}\left[\int\frac{dV}{8\pi}\;\epsilon_\delta^2\left|\mathbf{E}_1-\nu\;\frac{1}{c}\partial_t\widehat{\mathbf{A}}_1\right|^2\right]|_{\nu=0} &= \underbrace{-\epsilon_\delta^2\int\frac{dV}{4\pi}\;\mathbf{E}_1\cdot\frac{1}{c}\partial_t\widehat{\mathbf{A}}_1}_{\mathsf{Weak\ dynamics}}\\
&= \epsilon_\delta^2\int\frac{dV}{4\pi}\underbrace{\frac{1}{c}\;\partial_t\mathbf{E}_1\cdot\widehat{\mathbf{A}}_1}_{\mathsf{Dynamical\ term}} - \epsilon_\delta^2\int\frac{dV}{4\pi}\underbrace{\frac{1}{c}\;\partial_t\left(\mathbf{E}_1\cdot\widehat{\mathbf{A}}_1\right)}_{\mathsf{Noether's\ term}}.
\end{aligned}$$

the magnetic field contribution.

Next, we derive a magnetic contribution with $\mathbf{B}_1=\boldsymbol{\nabla}\times\mathbf{A}_1$:

$$\begin{aligned}
\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{mag}}}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1 &= -\frac{d}{d\nu}\left[\int\frac{dV}{8\pi}\left|\mathbf{B}_0+\epsilon_\delta\boldsymbol{\nabla}\times\left(\mathbf{A}_1+\nu\;\widehat{\mathbf{A}}_1\right)\right|^2\right]|_{\nu=0}\\
&= \underbrace{-\epsilon_\delta\int\frac{dV}{4\pi}\;\left(\mathbf{B}_0+\epsilon_\delta\mathbf{B}_1\right)\cdot\left(\boldsymbol{\nabla}\times\widehat{\mathbf{A}}_1\right)}_{\mathsf{Weak\ dynamics}}\\
&= -\epsilon_\delta\int\frac{dV}{4\pi}\left(\underbrace{\widehat{\mathbf{A}}_1\cdot\boldsymbol{\nabla}\times\left(\mathbf{B}_0+\epsilon_\delta\mathbf{B}_1\right)}_{\mathsf{Dynamical\ term}} - \underbrace{\boldsymbol{\nabla}\cdot\left[\left(\mathbf{B}_0+\epsilon_\delta\mathbf{B}_1\right)\times\widehat{\mathbf{A}}_1\right]}_{\mathsf{Noether's\ term}}\right).
\end{aligned} \tag{50}$$

3. Vlasov contributions

Let us now proceed with derivation of contribution from the Vlasov parts of the action functional.

The first variation of the linear Vlasov's part of the action writes as:

$$\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}} = -\left(\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta\mathcal{F}}\circ\delta\widehat{\mathcal{F}} + \frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta\phi_1}\circ\widehat{\phi}_1 + \frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1\right) \tag{51}$$

$$= -\int d^6Z\ dw\left(\ \mathcal{H}_1\ \delta\widehat{\mathcal{F}} + \mathcal{F}\ \left(\frac{\delta\mathcal{H}_1}{\delta\phi_1}\circ\widehat{\phi}_1 + \frac{\delta\mathcal{H}_1}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1\right)\ \right) \tag{52}$$

We now explicit each of those contributions with using the definition for the constrained Eulerian variation of the extended distribution function $\delta\mathcal{F}$ where $\mathcal{S}$ is the generating function, which does not affect dynamical part of variation, but defines the conservation law we derive via the Noether's method.

To explicit the first contribution we are using the expression for the constrained Eulerian variation of the Vlasov distribution (40), containing $\delta\widehat{\mathcal{F}}$. This expression can be further rewritten with using the Leibnitz rule for the Poisson bracket on the extended phase space such that the dynamical and the Noether contributions are:

$$\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta\mathcal{F}}\circ\delta\widehat{\mathcal{F}} = -\int d^6Z\ dw\ \mathcal{H}_1\ \{\mathcal{S},\widehat{\mathcal{F}}\}_{\mathsf{ext}} \tag{53}$$

$$= -\int d^6Z\ dw\ \underbrace{\{\mathcal{S}\mathcal{H}_1,\widehat{\mathcal{F}}\}_{\mathsf{ext}}}_{\text{Noether's term}} + \int d^6Z\ dw\ \mathcal{S}\ \underbrace{\{\mathcal{H}_1,\widehat{\mathcal{F}}\}_{\mathsf{ext}}}_{\text{Dynamical Vlasov}}$$

The first term here does not give any dynamical contributions and the second one leads toward derivation of Gyrokinetic Vlasov equation. The two remaining terms in (52) will contribute to the gyrokinetic Maxwell's equations via the polarization and magnetization.

Let us now consider the contributions to the first action functional variation from the nonlinear Vlasov part. The second order reduced dynamics in the functional (35) is associated with non-dynamical part of the distribution function F_0 only. it It will naturally lead to the contributions into the dynamical equations and does not provide any Noether's terms by construction.

$$\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}} = -\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}}}{\delta\phi_1}\circ\delta\widehat{\phi}_1 - \frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}}}{\delta\mathbf{A}_1}\circ\delta\widehat{\mathbf{A}}_1 = -\int d^6Z\ F_0\ \left(\frac{\delta H_2}{\delta\phi_1}\circ\widehat{\phi}_1 + \frac{\delta H_2}{\delta\mathbf{A}_1}\circ\widehat{\mathbf{A}}_1\right) \tag{54}$$

Let us now point out how the polarization and magnetization effects arise into the reduced Maxwell-Vlasov equations. Note that the reduced particle dynamics, contained into the Hamiltonians H_1 and H_2, is evaluated on the reduced phase space at the gyrocenter position $\mathbf{X}$, while

the electromagnetic potentials ϕ_1 and $\mathbf{A}_1$ are evaluated at the initial non-reduced space position $\mathbf{r}$. Therefore, the shift between both positions has to be systematically taken into account while calculating the functional derivatives. This fact naturally leads to the appearance of polarization and magnetization on the right hand side of the Gyrokinetic Poisson and Ampere equations.

As mentionned in the previous section, the first variation of the action functional can be rewritten in a form which contains two types of terms: those multiplied by the test functions $(\widehat{\phi}_1, \widehat{\mathbf{A}}_1)$ and the generating function S, and other terms, representing exact derivatives with respect to time and space variables. The first category of terms provides us equations of motion. The second one will be used later for derivation of the conservation laws via the Noether's theorem.

We proceed with explicit derivation of the polarization contributions from the reduced particles dynamics in the next section. In that section we continue with derivation of the reduced Maxwell-Vlasov equations in the implicit form as well as we sketch out here the general Noether procedure for the energy conservation law derivation.

B. Equations of motion: implicit weak form

We are now ready to write the reduced Maxwell-Vlasov system corresponding to the Eulerian second order action functional in an implicit form, i.e. without specifying expressions for the functional derivatives of the reduced Hamiltonians of the first and the second order, essentially representing polarization effects due to the dynamical reduction on the particle phase space. We proceed with the explicit derivation into the following section.

We start with writing implicit equations of motion in a *weak* form (i.e. applied on the test functions), which is essential for numerical implementation as well as for energy conservation derivation.

In order to write the *weak* form of the second order Gyrokinetic Maxwell - Vlasov equations we collect all the contributions to the first variation of the action functional accordingly to the (85).

We start with the gyrokinetic Poisson equation:

$$
\begin{aligned}
&0 = \frac{\delta \mathcal{A}^{\mathcal{E}}}{\delta \phi_1} \circ \widehat{\phi}_1 \Rightarrow \\
&\int dV \ \left(\mathbf{E}_1 \cdot \boldsymbol{\nabla} \widehat{\phi}_1 \right) = 4\pi \int dV \ dW \ \left(F_0 + \epsilon_\delta F_1 \right) \frac{\delta H_1}{\delta \phi_1} \circ \widehat{\phi}_1 + 4\pi \int dV \ dW \ F_0 \ \frac{\delta H_2}{\delta \phi_1} \circ \widehat{\phi}_1
\end{aligned}
\tag{55}
$$

and for the Ampere equation:

$$
\begin{aligned}
&0 = \frac{\delta \mathcal{A}^{\mathcal{E}}}{\delta \mathbf{A}_1} \circ \widehat{\mathbf{A}}_1 \Rightarrow \\
&\int dV \ \left[(\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \cdot \boldsymbol{\nabla} \times \widehat{\mathbf{A}}_1 + \frac{1}{c} \mathbf{E}_1 \cdot \partial_t \widehat{\mathbf{A}}_1 \right] = \\
&4\pi \int \ dV \ dW \ \ (F_0 + \epsilon_\delta F_1) \frac{\delta H_1}{\delta \mathbf{A}_1} \circ \widehat{\mathbf{A}}_1 + 4\pi \int \ dV \ dW \ F_0 \ \frac{\delta H_2}{\delta \mathbf{A}_1} \circ \widehat{\mathbf{A}}_1
\end{aligned}
\tag{56}
$$

Those equations can be rewritten in a strong form integrating by parts and then using the arbitrariness of the test functions: $\widehat{\phi}_1 \equiv \phi_1$ and $\widehat{\mathbf{A}}_1 \equiv \mathbf{A}_1$. By collecting terms multiplied by ϕ_1, we get the Poisson equation in a *strong* (conventional form)

$$
(\boldsymbol{\nabla} \cdot \mathbf{E}_1) \ = \ 4\pi \int \ dW \ e \, (F_0 + \epsilon_\delta F_1) \frac{\delta H_1}{\delta \phi_1} + 4\pi \int \ dW \ e \ F_0 \ \frac{\delta H_2}{\delta \phi_1}
\tag{57}
$$

The Ampere's equation is obtained from collecting terms multiplied by $\mathbf{A}_1$:

$$
\begin{aligned}
\left(\boldsymbol{\nabla} \times (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) - \epsilon_\delta \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t} \right) \ = \ & -4\pi \int \ dW \ e \, (F_0 + \epsilon_\delta F_1) \frac{\delta H_1}{\delta \mathbf{A}_1} \\
& - \ 4\pi \int \ dW \ e \ F_0 \ \frac{\delta H_2}{\delta \mathbf{A}_1}
\end{aligned}
\tag{58}
$$

Finally, with taking the test function $\widehat{\mathcal{F}} = \mathcal{F}$ the Vlasov equation is obtained in its extended phase space bracket form from (54):

$$
0 = \frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta \mathcal{F}} \circ \delta \widehat{\mathcal{F}} \Rightarrow \{\mathcal{H}_1, \mathcal{F}\}_{\mathsf{ext}} = 0
\tag{59}
$$

C. Noether method and energy conservation law

Noether's method in classical field theory is used to associate symmetries of action with conserved quantites. The general Noether's transport equation has the following form:

$$
\frac{\partial \mathsf{S}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = \delta \mathcal{L}^{\mathcal{E}}
\tag{60}
$$

where S is the Noether's current and $\mathbf{J}$ is the Noether's density; $\mathcal{L}^{\mathcal{E}}$ is Lagrangian density of Eulerian action:

$$
\mathcal{A}^{\mathcal{E}} \equiv \int dV \ \mathcal{L}^{\mathcal{E}}.
\tag{61}
$$

The variation $\delta \mathcal{L}^{\mathcal{E}}$ is defined accordingly to the conservation law we are deriving.

By collecting the Noether's terms, which we have derived in the previous section, while rewriting equations of motion in a strong form, we can write general expressions for S and $\mathbf{J}$ suitable for conservation laws derivation:

$$\mathsf{S} = -\frac{\epsilon_\delta^2}{4\pi}\frac{1}{c}\left(\mathbf{E}_1 \cdot \widehat{\mathbf{A}}_1\right) + \int dW\ dw\ \mathcal{F}\ \mathcal{S} \tag{62}$$

$$\mathbf{J} = -\frac{\epsilon_\delta^2}{4\pi}\ \mathbf{E}_1\ \widehat{\phi}_1 + \frac{\epsilon_\delta}{4\pi}\left[(\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \widehat{\mathbf{A}}_1\right] + \int dW\ dw\ \mathcal{F}\{\mathbf{X}, H_1\}\mathcal{S} \tag{63}$$

The last term in the expression for Noether's current S and Noether's density $\mathbf{J}$ is obtained from Vlasov part of action functional, its explicit derivation from the expressions (54) are summarized in (C).

In this section we will concentrate on the energy conservation derivation only. We leave momentum conservation derivation for further work.

The energy conservation is derived from performing infinitesimal time translations $t \to t + \delta t$ on the Eulerian action $\mathcal{A}^{\mathcal{E}}$. The explicit expression for the corresponding generating function $\mathcal{S}$, which also the defines the constrained variations of Vlasov field is given by $\mathcal{S} = -w\ \delta t$. Then, the expressions of electromagnetic field and Lagrangian density variations are defined as:

$$\delta\phi_1 = -\delta t\ \frac{\partial \phi_1}{\partial t} \tag{64}$$

$$\delta\mathbf{A}_1 = -\delta t\ \frac{\partial \mathbf{A}_1}{\partial t} = c\ \delta t\,(\mathbf{E}_1 + \boldsymbol{\nabla}\phi_1) \tag{65}$$

$$\delta\mathcal{L}^{\mathcal{E}} = -\delta t\left(\frac{\partial \mathcal{L}^{\mathcal{E}}}{\partial t} - \frac{\partial' \mathcal{L}^{\mathcal{E}}}{\partial t'}\right), \tag{66}$$

where $\mathbf{X}$ is the gyrocenter poistion; the ∂'_t means time derivative with respect to the background fields only. In general, the background non-dynamical fields $\mathbf{B}_0$ and F_0 could explicitely depdend on time. In this case their explicit time-dependencies should be removed from the conservation law. In our case, both of those fields are explicitly time-independent, so the variation of Lagrangian density is simply given by $\delta\mathcal{L}^{\mathcal{E}} = -\delta t\ \partial_t \mathcal{L}^{\mathcal{E}}$.

Remark that, only Maxwellian part of $\mathcal{A}^{\mathcal{E}}$ and the Vlasov part associated with non-dynamical part of the distribution function will contribute to $\delta\mathcal{L}^{\mathcal{E}}$, because formally the Vlasov part $\int d^8Z\ \mathcal{F}\ \mathcal{H} \equiv 0$ for $\mathcal{H} \equiv \delta(w - H)$.

In the following section, we derive explicit expression for energy density, which contains FLR terms up to the second order.

The Lagrangian density time derivative is given by:

$$\delta\mathcal{L}^{\mathcal{E}} = -\delta t\ \frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(\epsilon_\delta^2 \left|\,\mathbf{E}_1\,\right|^2 - \left|\,\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1\,\right|^2\right) - \int dW\ F_0\ H_2\right]. \tag{67}$$

It means that in the case of the energy conservation derivation through the infinitesimal time translations, the Maxwell part of the action as well as the truncated non-dynamical Vlasov part, contributes to the energy density.

We can now write an implicit expression for the energy density $\mathcal{E}$, which corresponds to the Eulerian action (35). With choosing the test functions, corresponding to the conservation law we are about to derive: $\widehat{\phi}_1 = \delta\phi_1$ and $\widehat{\mathbf{A}}_1 = \delta\mathbf{A}_1$, we get:

$$\mathcal{E} = -\frac{1}{8\pi}\left(\epsilon_\delta^2 |\mathbf{E}_1|^2 + |\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1|^2\right) - \epsilon_\delta^2 \int dW \, (F_1 \; H_1 + F_0 \; H_2) \tag{68}$$

$$- \frac{\epsilon_\delta^2}{4\pi} \, \mathbf{E}_1 \cdot \boldsymbol{\nabla}\phi_1 \tag{69}$$

$$\mathbf{J} = \frac{\epsilon_\delta}{4\pi} \, (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \mathbf{E}_1 - \int dW \; F_1 \dot{\mathbf{X}}^{(1)} \tag{70}$$

$$+ \frac{\epsilon_\delta^2}{4\pi} \left[\, \mathbf{E}_1 \, \frac{\partial \phi_1}{\partial t} \; + c \, (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \boldsymbol{\nabla}\phi_1 \, \right] \tag{71}$$

One more manipulation should be done before we proceed with an explicit derivation for some particular choice of the reduced particle model with a particular choice of H_2.

The terms (69) and (71) can be rewritten with using the weak form of the equations of motion (56) and (56), derived for this model.

With using the test function $\widehat{\phi}_1 \equiv \phi_1$ in the Poisson equation (57) and $\widehat{\mathbf{A}}_1 \equiv \boldsymbol{\nabla}\phi_1$ in (58) we introduce polarization and magnetization effects into the expression for the energy density $\mathcal{E}$ and the energy flux $\mathbf{J}$:

$$\mathcal{E} = -\frac{1}{8\pi}\left(\epsilon_\delta^2 |\mathbf{E}_1|^2 + |\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1|^2\right) - \epsilon_\delta^2 \int dW \, (F_1 \; H_1 + F_0 \; H_2) \tag{72}$$

$$- \int dW \; \epsilon_\delta \, (F_0 + \epsilon_\delta F_1) \, \frac{\delta H_1}{\delta \phi_1} \circ \phi_1 \tag{73}$$

$$\mathbf{J} = \frac{\epsilon_\delta}{4\pi} \, (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \mathbf{E}_1 - \epsilon_\delta \int dW \; F_1 \dot{\mathbf{X}}^{(1)}_{\mathrm{gy}} \tag{74}$$

$$- \epsilon_\delta \int dW \;\; (F_0 + \epsilon_\delta F_1) \, \frac{\delta H_1}{\delta \mathbf{A}_1} \circ \boldsymbol{\nabla}\phi_1 - \epsilon_\delta \int dW \; F_0 \; \frac{\delta H_2}{\delta \mathbf{A}_1} \circ \boldsymbol{\nabla}\phi_1 \tag{75}$$

In the following section we proceed with derivation of the explicit expressions for equations of motion and the energy conservation laws.

IV. EULERIAN SECOND ORDER ACTION FUNCTIONAL: EXPLICIT DERIVATION

The aim of this section is to provide an explicit form of the reduced Maxwell-Vlasov equations in their *weak* form (56, 56) and in their *strong* (57, 58) form.

To start the explicit derivation we are implementing the standard approximation of the Gyrokinetic theory concerning the electromagnetic fields.

First, we take into account the low frequency approximation for the electric field: $\mathbf{E}_{1\perp} \equiv -\boldsymbol{\nabla}_\perp \phi_1$ for the perpendicular component and $E_{1\parallel} = -\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\phi_1 - c^{-1}\partial_t A_{1\parallel}$ for the parallel one. Therefore, the parallel electric field component is of the next order of smallness with respect to the gyrocenter parameter ϵ_δ than the perpendicular one: $|E_{1\parallel}| \sim \epsilon_\delta |\mathbf{E}_{1\perp}|$.

In the same time, we suppose that the magnetic perturbation is given by:

$$\mathbf{B}_1 = \nabla \times (\widehat{\mathbf{b}}\, A_{1\parallel}), \tag{76}$$

i.e. the magnetic potential perturbation has only a parallel component. We keep here the full expression of the perturbed magnetic field (76), which respects the Maxwell constraint $\boldsymbol{\nabla} \cdot \mathbf{B}_1 = 0$ and therefore conserves the Liouville theorem necessary for the phase space volume conservation on the reduced phase space. The latter leads to appearance of magnetic curvature terms in the Ampère's law and the conservations laws. We also remark that, taking into account only parallel fluctuations of the perturbed magnetic potential, leads to the derivation of only parallel component of Ampère's law from the variational principle.

Finally, in what concerns the particle dynamics a choice of the reduced model is realized accordingly to the gyrocenter Hamiltonian (29). The detailed derivation of that expression is done in sec. (II D 2).

The H_1 is the first order correction to the general gyrocenter Hamiltonian (29):

$$\begin{aligned} H_1 &= \epsilon_\delta \left(e \left\langle \phi_1(\mathbf{X} + \boldsymbol{\rho}_0) \right\rangle - \frac{1}{m}\frac{e}{c}\, p_z \, \left\langle A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0) \right\rangle \right) \\ &\equiv \epsilon_\delta \left(e \left\langle \phi_{1gc} \right\rangle - \frac{1}{m}\frac{e}{c}\, p_z \, \left\langle A_{1\parallel gc} \right\rangle \right) \end{aligned} \tag{77}$$

While the H_2 is represented by the $\mathcal{O}(\epsilon_\delta^2)$ corrections to the (29): we truncate the gyro averaging up to the second order in guiding-center FLR corrections for the squared parallel magnetic potential: $\left\langle A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0)^2 \right\rangle$:

$$\begin{aligned} \left\langle A_{1\parallel}(\mathbf{X} + \boldsymbol{\rho}_0)^2 \right\rangle &= \left\langle A_{1\parallel}(\mathbf{X})^2 \right\rangle + \rho_0^2 \left\langle \left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} A_{1\parallel}(\mathbf{X}) \right) \left(\widehat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla} A_{1\parallel}(\mathbf{X}) \right) \right\rangle \\ &+ \rho_0^2 \, A_{1\parallel} \left\langle \widehat{\boldsymbol{\rho}}\, \widehat{\boldsymbol{\rho}} \right\rangle : \boldsymbol{\nabla}\boldsymbol{\nabla} A_{1\parallel}. \end{aligned} \tag{78}$$

Next, with taking into account the expression for the gyroaveraged tensor (27) and the expression for the lower order guiding-center displacement (17), we can write the explicit expression for the second order Hamiltonian H_2:

$$H_2 = \frac{e^2}{2mc^2} A_{1\|}^2 + \frac{\mu}{2B} \left|\nabla_\perp A_{1\|}\right|^2 + \frac{1}{2}\frac{\mu}{B} A_{1\|} \nabla_\perp^2 A_{1\|} - \frac{mc^2}{2B^2} \left|\nabla_\perp \phi_1 - \frac{p_z}{mc} \nabla_\perp A_{1\|}\right|^2 \tag{79}$$

The first and the second terms are the first order guiding-center FLR corrections to the averaged gyrocenter magnetic potential, and the third term is the second order guiding-center FLR contribution.

The last term here represents the lowest order gyrocenter polarization correction, associated with the gyro averaged gyrocenter displacement $\boldsymbol{\rho}_1$ (28). The latter is related to the gradient of the electromagnetic potential in general Gyrokinetic theory. We emphasize that most of the physical models consider only its electrostatic part. In the sec. (V) we make a comparison between this full second order model and the ORB5 model, which contains only the electrostatic part of the polarization.

Remark that the electromagnetic part of the polarization contains only the FLR corrections of the first order, which does not contradict to the general gyrocenter ordering consistency.

A. Dynamical and Noether's terms

Taking into account assumptions on the electromagnetic fields as well as on truncated particle's dynamics discussed in the section above, we can write an explicit expression for the Eulerian action:

$$\mathcal{I}_\|^{\mathcal{E}} \left[\phi_1, A_{1\|}, \mathcal{F}\right] \equiv \int_{t_1}^{t_2} \mathcal{A}_\|^{\mathcal{E}} \left[\phi_1, A_{1\|}, \mathcal{F}\right] \, dt \tag{80}$$

$$= \int \frac{dV \, dt}{8\pi} \left(\epsilon_\delta^2 \left|\boldsymbol{\nabla}_\perp \phi_1\right|^2 - \left|\mathbf{B}_0 + \epsilon_\delta \, \boldsymbol{\nabla} \times \left(A_{1\|} \widehat{\mathbf{b}}\right)\right|^2\right) \tag{81}$$

$$- \int d^8 Z \, \mathcal{F} \left[H_0 + \epsilon_\delta \, e \left\langle \phi_{1\mathrm{gc}} \right\rangle - \epsilon_\delta \, e \, \frac{p_z}{mc} \left\langle A_{1\|gc} \right\rangle - w\right] \tag{82}$$

$$+ \frac{\epsilon_\delta^2}{2} \int d^8 Z \, \frac{mc^2}{B^2} \, F_0 \, \left|\boldsymbol{\nabla}_\perp \phi_1 - \frac{p_z}{mc} \boldsymbol{\nabla}_\perp A_{1\|}\right|^2 \tag{83}$$

$$- \frac{\epsilon_\delta^2}{2} \int d^8 Z \, F_0 \, \left(\frac{1}{m}\left(\frac{e}{c}\right)^2 A_{1\|}^2 + \frac{\mu}{B}\left|\boldsymbol{\nabla}_\perp A_{1\|}\right|^2 + \frac{\mu}{B} A_{1\|} \boldsymbol{\nabla}_\perp^2 A_{1\|}\right), \tag{84}$$

where we have used the definition of the extended Vlasov function (36).

We are following the same procedure of the first variation calculation as in the previous section within the implicit derivation.

$$\delta\mathcal{I}_\|^{\mathcal{E}} \equiv \int_{t_1}^{t_2} \delta\mathcal{A}^{\mathcal{E}} \left[\phi_1, A_{1\|}, \mathcal{F}\right] dt = \int_{t_1}^{t_2} \left(\frac{\delta \mathcal{A}_\|^{\mathcal{E}}}{\delta \phi_1} \circ \widehat{\phi}_1 + \frac{\delta \mathcal{A}_\|^{\mathcal{E}}}{\delta A_{1\|}} \circ \widehat{A}_{1\|} + \frac{\delta \mathcal{A}_\|^{\mathcal{E}}}{\delta \mathcal{F}} \circ \delta\widehat{\mathcal{F}}\right) dt. \tag{85}$$

We summarize the details of that calculation into the appendix A. Here we provide the final form of the gyrokinetic Maxwell-Vlasov equations in *weak* and *strong* form as well as the expression for the conserved energy density.

B. Equations of motion

1. Gyrokinetic Vlasov equation

The Vlasov equation follows from the variational principle in the form of an exact derivative accordingly to the (59). This is equivalent to the statement that the Vlasov equation is reconstructed from the first order gyrocenter characteristics (31). We explicit this equation as follows:

$$\frac{\partial F_1}{\partial t} \;=\; -\{F_1, H_0\}_{\rm gc} - \{F_0, H_1\}_{\rm gc} - \epsilon_\delta \{F_1, H_1\}_{\rm gc},$$

where the two first terms represent linear drive in the system (first term: coupling between background dynamics and dynamical part of the Vlasov field; second term: coupling of the background (non-dynamical) distribution with the first order fluctuations), the last term represents the nonlinear coupling between the dynamical part of the Vlasov field with the first order Hamiltonian.

2. Gyrokinetic Poisson equation

The Poisson equation in a *weak* form is given by:

$$\begin{aligned} 0 \;&=\; \frac{\delta \mathcal{A}_\parallel{}^{\mathcal{E}}}{\delta \phi_1} \circ \widehat{\phi}_1 \Rightarrow \\ \epsilon_\delta \frac{1}{4\pi} \int dV\; \boldsymbol{\nabla}_\perp \phi_1 \cdot \boldsymbol{\nabla}_\perp \widehat{\phi}_1 \;&=\; \epsilon_\delta \int dV\; dW \;\; \left(\frac{mc^2}{B^2}\; F_0\right) \left[\boldsymbol{\nabla}_\perp \phi_1 - \frac{p_z}{mc} \boldsymbol{\nabla}_\perp A_{1\parallel}\right] \cdot \boldsymbol{\nabla}_\perp \widehat{\phi}_1 \\ &-\; e \int\; dV\; dW \;\; (F_0 + \epsilon_\delta F_1) \;\left\langle \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\; \widehat{\phi}_1 \right\rangle \end{aligned} \tag{86}$$

Here we define the guiding-center gyro-averaging operator $\mathcal{J}_0^{\rm gc}$ as follows:

$$\left\langle \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\; \widehat{\phi}_1 \right\rangle \;\equiv\; \int_0^{2\pi} d\theta\; \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\; \widehat{\phi}_1(\mathbf{r}) \tag{87}$$

$$=\; \int_0^{2\pi} d\theta\; \widehat{\phi}_1\left(\mathbf{X} + \boldsymbol{\rho}_0\right) \equiv \mathcal{J}_0^{\rm gc}\left(\widehat{\phi}_1\right) \tag{88}$$

And therefore,

$$\int\; dV\; dW\;\; (F_0 + \epsilon_\delta F_1)\;\left\langle \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\; \widehat{\phi}_1 \right\rangle \equiv \int\; dV\; dW\;\; (F_0 + \epsilon_\delta F_1)\;\; \mathcal{J}_0^{\rm gc}\left(\widehat{\phi}_1\right). \tag{89}$$

we can make it acting on the distribution function $F_0 + \epsilon_\delta F_1$ rather than on the test function $\widehat{\phi}_1$ as follows:

$$0 = \left[-\epsilon_\delta \ \frac{1}{4\pi} \boldsymbol{\nabla}_\perp^2 \phi_1 + \epsilon_\delta \ \int \ dp_z \ d\mu \ \boldsymbol{\nabla}_\perp \cdot \left[\frac{mc^2}{B^2} B_\parallel^* \ F_0 \ \boldsymbol{\nabla}_\perp \left(\phi_1 - \frac{p_z}{mc} A_{1\parallel} \right) \right] \right.$$
$$\left. - e \int \ dp_z \ d\mu \ B_\parallel^* \ \mathcal{J}_0^{\mathrm{gc}\dagger} \left(F_0 + \epsilon_\delta F_1 \right) \right] \tag{90}$$

This is the *strong* form of the Poisson equation. Remark that the requirement for the gyroaveraging operator being hermitian, i.e. $\mathcal{J}_0^{\mathrm{gc}} = \mathcal{J}_0^{\dagger \mathrm{gc}}$ is not necessary in case of the finite-element discretisation performed for construction of PIC code, because in that case we are discretising equations in their *weak* form and we do not need to shift the gyroaveraging operator from the test function $\widehat{\phi}_1$ to the distribution function. However, an example of the hermitian gyroaveraging operator can be found in [14].

3. Gyrokinetic Ampère equation

The *weak* formulation of the Ampère equation writes:

$$\begin{aligned}
0 \ &= \ \frac{\delta \mathcal{A}_\parallel{}^{\mathcal{E}}}{\delta \phi_1} \circ \widehat{A}_{1\parallel} \Rightarrow \\
0 \ &= \ - \int \frac{dV}{4\pi} \ \epsilon_\delta \left| \mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1 \right| \cdot \boldsymbol{\nabla} \times \left(\widehat{A}_{1\parallel} \widehat{\mathbf{b}} \right) \\
&- \ \epsilon_\delta^2 \int dV \ dW \ \frac{mc^2}{B^2} F_0 \left| \boldsymbol{\nabla}_\perp \phi_1 - \frac{p_z}{mc} \boldsymbol{\nabla}_\perp A_{1\parallel} \right| \cdot \boldsymbol{\nabla}_\perp \widehat{A}_{1\parallel} - \epsilon_\delta^2 \int dV \ dW \ F_0 \ \left(\frac{e^2}{mc^2} A_{1\parallel} \widehat{A}_{1\parallel} \right. \\
&+ \ \left. \frac{\mu}{B} \left[|\boldsymbol{\nabla}_\perp A_{1\parallel}| \cdot \boldsymbol{\nabla}_\perp \widehat{A}_{1\parallel} + A_{1\parallel} \boldsymbol{\nabla}_\perp^2 \widehat{A}_{1\parallel} + \widehat{A}_{1\parallel} \boldsymbol{\nabla}_\perp^2 A_{1\parallel} \right] \right) \\
&+ \ \epsilon_\delta \int \ dV \ dW \ \left(F_0 + \epsilon_\delta F_1 \right) \ \frac{p_z}{mc} \mathcal{J}_0^{\mathrm{gc}} \left(\widehat{A}_{1\parallel} \right).
\end{aligned} \tag{91}$$

We remark that the choice of a parallel component of perturbed magnetic potential $A_{1\parallel}$ naturally leads toward derivation of only a parallel component of the *strong* Gyrokinetic Ampère equation:

$$\begin{aligned}
0 = &\left[\frac{1}{4\pi} \widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \left(\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1 \right) - \epsilon_\delta \ \int \ d\mu \ dp_z \boldsymbol{\nabla}_\perp \cdot \left[\frac{mc}{B^2} \ B_\parallel^* \ F_0 \ \frac{p_z}{m} \boldsymbol{\nabla}_\perp \left(\phi_1 - \frac{p_z}{mc} A_{1\parallel} \right) \right] \right. \\
&+ \epsilon_\delta \ \frac{e^2}{mc^2} \int \ dW \ F_0 \ A_{1\parallel} + \epsilon_\delta \int \ d\mu \ dp_z \nabla_\perp \ \left(\frac{\mu}{B} B_\parallel^* \ F_0 \right) \ \nabla_\perp A_{1\parallel} \\
&\left. - \frac{\epsilon_\delta^2}{2} \int \ d\mu \ dp_z \ \nabla_\perp^2 \left(\frac{\mu}{B} \ B_\parallel^* F_0 \right) A_{1\parallel} + \int dW \ \frac{e \ p_z}{mc} \ \mathcal{J}_0^{\mathrm{gc}\dagger} \left(F_0 + \epsilon_\delta F_1 \right) \right]
\end{aligned} \tag{92}$$

C. Conserved energy density

1. Final expression for conserved energy density

By substituting variations associated with time translations (64, 65, 66) into the general expressions for the Noether density S, using the equations of motion associated with $\mathcal{A}^{\mathcal{E}}$, derived into the previous section, we get the expression for energy density into the implicit form:

$$\begin{aligned}\mathcal{E} \;=\; & \int \frac{dV}{8\pi}\left(\epsilon_\delta^2\left|\mathbf{E}_1\right|^2+\left|\mathbf{B}_0+\epsilon_\delta\mathbf{B}_1\right|^2\right)+\int dV\;dW\;F_0\left(H_2+\epsilon_\delta^2\;\frac{mc^2}{B^2}\left|\nabla_\perp\phi_1\right|^2\right)\\ & +\int dV\;dW\;\;\left(F_0+\epsilon_\delta F_1\right)\left(H_0+\epsilon_\delta H_1-\epsilon_\delta\;e\left\langle\phi_{1gc}\right\rangle\right).\end{aligned} \tag{93}$$

This expression contains information about the gyrocenter dynamical reduction up to the second order, i.e. ϵ_δ^2 and keeps the FLR corrections at all orders.

The explicit expression for the energy density with second order FLR corrections into the nonlinear terms is obtained by substituting the explicit expressions into (93):

$$\begin{aligned}\mathcal{E}_2=\int dV\;dW\;\;\left(F_0+\epsilon_\delta F_1\right)\left(H_0-\epsilon_\delta\;e\frac{p_z}{m}\left\langle A_{1\parallel gc}\right\rangle\right)\\ +\frac{\epsilon_\delta^2}{2}\int dV\;dW\;F_0\left(\frac{e^2}{c^2}\frac{1}{m}A_{1\parallel}^2+\frac{\mu}{B}\left(\nabla_\perp A_{1\parallel}\right)^2+\frac{\mu}{B}A_{1\parallel}\nabla_\perp^2A_{1\parallel}\right)\\ +\frac{\epsilon_\delta^2}{2}\int dV\;dW\;F_0\;\frac{mc^2}{B^2}\left(\left|\nabla_\perp\phi_1\right|^2-\left(\frac{p_z}{mc}\right)^2\left|\nabla_\perp A_{1\parallel}\right|^2\right)\\ +\frac{1}{8\pi}\int dV\left(\epsilon_\delta^2|\nabla_\perp\phi_1|^2+|\mathbf{B}_0+\epsilon_\delta\mathbf{B}_1|^2\right).\end{aligned} \tag{94}$$

We remark, that in the electromagnetic case, there is a part of energy provided by the magnetic field (background and fluctuating) in the system. Therefore the field energy contribution can not be completely removed via using the quasineutrality approximation, as it was previously possible in the electrostatic case. The field part of energy should be then included into the code diagnostics.

V. EULERIAN VARIATIONAL PRINCIPLE FOR THE ORB5 CODE MODEL

In a previous section we have explicitly derived equations of motion and the conserved energy density corresponding to the Eulerian action functional (80), which contains up to the $\mathcal{O}(\epsilon_\delta^2)$ terms together with the second order FLR corrections. In this section we rewrite the second order Eulerian variational functional in a more compact form and then perform on it all necessary approximations in order to be able to access throughout the first variation calculation the gyrokinetic Maxwell-Vlasov system of equations currently implemented into the ORB5 code. In the same time, we aim

to compare the conserved energy density corresponding to the variational principle with diagnostics of the code. Our main goal here is to make a comparison between the gyrokinetic Maxwell-Vlasov models issued from a different first principle derivations.

A. Second order action functionals

$$\mathcal{I}^{\mathcal{E}}_{\text{full}} \equiv \int_{t_1}^{t_2} \mathcal{A}^{\mathcal{E}}_{\mathsf{full}}\, dt = \int \frac{dV}{8\pi}\, \left(\epsilon_\delta^2 \left|\boldsymbol{\nabla}_\perp \phi_1\right|^2 - \left| \mathbf{B}_0 + \epsilon_\delta\, \boldsymbol{\nabla} \times \left(A_{1\parallel} \widehat{\mathbf{b}} \right) \right|^2 \right) \\ - \sum_{\text{sp}} \int d^8 Z\, \mathcal{F}\, \mathcal{H}_1 - \sum_{\text{sp}} \int d^6 Z\, F_0\, H_2^{\text{full}} \tag{95}$$

To get the ORB5 code model, two physical approximations should be performed on this full second order action principle: one on the Maxwellian part of that action functional and the another one on this particle's part.

1. The quasi neutrality approximation

Let us start with considering the most common physical reduction: the quasi neutrality approximation, which is implemented into the ORB5 model. Considered at the action functional level, this approximation can be made without making loose of the energetic consistency in further derivation.

The quasi-neutrality approximation allows to neglect the $|\mathbf{E}_1|^2$ term in Maxwell part of the Eulerian action. The latter term is usually ordered small comparing to the second order polarization term, contained in H_2 proportional to $\boldsymbol{\nabla}_\perp \phi_1$. As we have already mentioned in previous section, standard gyrokinetic ordering pushes parallel component of electric field at the next order of smallness comparing to its perpendicular component $|E_{1\parallel}| \sim \epsilon_\delta |\mathbf{E}_{1\perp}|$. The Eulerian action (96) does not contains parallel component of electric field. In addition to that we are now taking into account one of the characteristic spatial scales separation in fusion plasmas, resulting from the fact that the ion sound Larmor radius $\rho_s^2 \equiv \frac{2k_B T_e mc^2}{e^2 B^2}$ is larger than the Debye length $\lambda_d^2 \equiv \frac{k_B T_e}{4\pi n e^2}$:

$$\frac{\rho_s^2}{\lambda_d^2} = \frac{8\pi n m c^2}{B^2} = \frac{c^2}{v_A^2} \gg 1, \tag{96}$$

where v_A is the Alfvèn velocity.

In electrostatic approximation:

$$\int \frac{dV}{8\pi} |\boldsymbol{\nabla}_\perp \phi_1|^2 + \int dV\, dW\, F_0\, \frac{mc^2}{2B^2} |\boldsymbol{\nabla}_\perp \phi_1|^2 = \frac{1}{8\pi} \int dV\, \left(1 + \frac{\rho_s^2}{\lambda_d^2} \right) |\boldsymbol{\nabla}_\perp \phi_1|^2. \tag{97}$$

Therefore, the electrostatic contribution to the Maxwell's part of the action functional can be omitted.

2. Perturbed magnetic field approximation

The next approximation which has to be made, concerns again the Maxwellian part of the action: it is about the perturbed part of the magnetic field.

Most of the physical models omit curvature contributions to the perpendicular part of magnetic field perturbation $\mathbf{B}_1$, i.e. the term $\mathbf{B}_\perp = \widehat{\mathbf{b}} \times \nabla A_\parallel$. The latter leads to violation of the divergence free Maxwell constraint up to the second order in ϵ_B, referred earlier as the small parameter related to the background fields non-uniformities.

3. Particle dynamics approximation

The last approximation should be performed on the particle's level of dynamics.

To get the ORB5 code model, which at the moment does not take into account coupling between the reduced Poisson and Ampère equations, the second order Hamiltonian H_2 should not contain any "mixed" electromagnetic potential perturbation.

This is why the ORB5 model uses linear polarization approximation:

$$H_2^{\mathsf{ORB5}} \equiv \frac{e^2}{2mc^2} \left\langle A_{1\parallel gc} \right\rangle^2 - \frac{mc^2}{2B^2} \left| \nabla_\perp \phi_1 \right|^2 , \tag{98}$$

which is different to H_2 issued from the direct derivation within the Gyrokinetic reduction (79).

The final expression of the action functional providing the system of the Maxwell-Vlasov equations corresponding to the ORB5 model is:

$$\begin{aligned} \mathcal{I}^{\mathcal{E}}_{\mathrm{ORB5}} \;\equiv\; & \int_{t_1}^{t_2} \mathcal{A}^{\mathcal{E}}{}_{\mathsf{ORB5}}\; dt = -\frac{\epsilon_\delta^2}{8\pi} \int dV \left| \boldsymbol{\nabla}_\perp A_{1\parallel} \right|^2 \\ & - \sum_{\mathrm{sp}} \int d^8 Z\; \mathcal{F}\; \mathcal{H}_1 - \sum_{\mathrm{sp}} \int d^6 Z\; F_0\; H_2^{\mathrm{ORB5}} \end{aligned} \tag{99}$$

B. ORB5 Maxwell-Vlasov model

In order to make a comparison between the system of the gyrokinetic Maxwell-Vlasov equations issued from the full second order derivation and those currently implemented into the ORB5 codes

we truncate the FLR decomposition in (98) for the gyroaveraged magnetic potential $\langle A_{1\|gc}\rangle$ at the second order as in ($A9$).

Using the Taylor expansion in the vicinity of the guiding-center position $\mathbf{X}$ and keeping the guiding-center polarization corrections up to the second order:

$$A_{1\|gc} = A_{1\|}(\mathbf{X}) + \boldsymbol{\rho}_0 \cdot \boldsymbol{\nabla} A_{1\|}(\mathbf{X}) + \frac{1}{2}\boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \boldsymbol{\nabla}\boldsymbol{\nabla} A_{1\|}(\mathbf{X}), \tag{100}$$

where the lowest order guiding-center displacement $\boldsymbol{\rho}_0$ is given by (17).

The first term in (98) writes:

$$\langle A_{1\|gc}\rangle^2 = \left(A_{1\|} + \frac{1}{2}\langle \boldsymbol{\rho}_0\boldsymbol{\rho}_0\rangle : \boldsymbol{\nabla}\boldsymbol{\nabla} A_{1\|}\right)^2 = A_{1\|}^2 + m\left(\frac{c}{e}\right)^2 \frac{\mu}{B} A_{1\|}\, \boldsymbol{\nabla}_\perp^2 A_{1\|}, \tag{101}$$

while the (79) contains additional first order FLR contribution:

$$\begin{aligned}\left\langle A_{1\|gc}^2\right\rangle = & \left\langle \left(A_{1\|}^2 + \boldsymbol{\rho}_0 \cdot \boldsymbol{\nabla} A_{1\|} + \frac{1}{2}\boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \boldsymbol{\nabla}\boldsymbol{\nabla} A_{1\|}\right)^2 \right\rangle \\ & = A_{1\|}^2 + m\left(\frac{c}{e}\right)^2 \frac{\mu}{B}\left(\boldsymbol{\nabla}_\perp A_{1\|}\right)^2 + m\left(\frac{c}{e}\right)^2 \frac{\mu}{B} A_{1\|}\, \boldsymbol{\nabla}_\perp^2 A_{1\|}\end{aligned} \tag{102}$$

In what concerns the second terms into the expressions (79) and (98), the first one contains electromagnetic corrections, while the second expression is restricted to the electrostatic corrections only.

As we have already mentioned above, the presence of the electromagnetic contribution in H_2, results in coupling between the reduced Poisson (quasi-neutrality) and Ampère equations.

Note that coupling of gyrokinetic Maxwell equations can be of particular interest for further numerical studies with canonical Maxwellian initialization (asymmetric background distribution). Such an implementation has a particular interest for energetic particles investigation, for example.

Let us now analyze in details the corresponding equations of motion.

C. Quasineutrality equation

We start with comparing the quasi neutrality equations. First, the one which follows from the second order Eulerian action:

$$-\ \epsilon_\delta \sum_{\mathrm{sp}} \int dW \frac{1}{B_\|^*}\, \boldsymbol{\nabla}_\perp \left[B_\|^* F_0 \frac{mc^2}{B^2} \boldsymbol{\nabla}_\perp \left(\phi_1 - \frac{p_z}{mc} A_{1\|}\right)\right] = \sum_{\mathrm{sp}} e \int dW\ \mathcal{J}_0^{\mathrm{gc}\dagger}\left(F_0 + \epsilon_\delta F_1\right) \tag{103}$$

In the same time, the quasi neutrality condition, following from the ORB5 action writes:

$$-\sum_{\mathrm{sp}} \int dW \frac{1}{B_\|^*} \boldsymbol{\nabla}_\perp \left[B_\|^* F_0 \frac{mc^2}{B^2} \boldsymbol{\nabla}_\perp \phi_1\right] = \sum_{\mathrm{sp}}\ e \int dW\ \mathcal{J}_0^{\mathrm{gc}\dagger} F_1 \tag{104}$$

D. Ampère's equation

Taking into account the same approximation as in the ORB5 model with $\mathbf{B}_1 = \widehat{\mathbf{b}} \times \boldsymbol{\nabla} A_{1\parallel}$ and $\widehat{\mathbf{b}} \times \widehat{\mathbf{b}} \times \boldsymbol{\nabla} A_{1\parallel} = -\boldsymbol{\nabla}_\perp^2 A_{1\parallel}$, the Ampère equation, which follows from the Eulerian action:

$$\begin{aligned}\frac{1}{4\pi}\boldsymbol{\nabla}_\perp^2 A_{1\parallel} \;=\; & -\int dW\; \frac{ep_z}{mc}\, \mathcal{J}_0^{\mathrm{gc}\dagger} F_1 + \int dW\; \frac{e^2}{mc^2}\left(A_{1\parallel} F_0\right) + \frac{1}{2}\int dW\; \frac{1}{B_\parallel^*}\boldsymbol{\nabla}_\perp^2\left(B_\parallel^* \frac{\mu}{B} F_0\right) A_{1\parallel} \\ & - \int dW\; \frac{1}{B_\parallel^*}\boldsymbol{\nabla}_\perp\left(B_\parallel^* \frac{\mu}{B} F_0\right)\boldsymbol{\nabla}_\perp A_{1\parallel} - \int dW\; \left(F_0 \frac{c\, p_z}{B^2}\right)\boldsymbol{\nabla}_\perp\left[\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel} - \boldsymbol{\nabla}_\perp \phi_1\right] \\ & - \int dW\; \frac{1}{B_\parallel^*}\boldsymbol{\nabla}_\perp\left(F_0 B_\parallel^* \frac{c\, p_z}{B^2}\right)\cdot\boldsymbol{\nabla}_\perp\left[\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel} - \boldsymbol{\nabla}_\perp \phi_1\right] \end{aligned} \tag{105}$$

while the Ampere equation, which follows from the physical action is:

$$\begin{aligned}\frac{1}{4\pi}\boldsymbol{\nabla}_\perp^2 A_{1\parallel} \;=\; & -\int dV\; dW\; \frac{ep_z}{mc}\, \mathcal{J}_0^{\mathrm{gc}\dagger} F_1 + \int dW\; \frac{e^2}{mc^2}\left(A_{1\parallel} F_0\right) + \frac{1}{2}\int dW\; \frac{1}{B_\parallel^*}\boldsymbol{\nabla}_\perp^2\left(B_\parallel^* \frac{\mu}{B} F_0\right) A_{1\parallel} \\ & + \int dW \frac{\mu}{B} F_0\; \boldsymbol{\nabla}_\perp^2 A_{1\parallel}. \end{aligned} \tag{106}$$

We can see that Ampère's law issued from the physical Eulerian action neglects coupling with electrostatic potential ϕ_1. Terms proportional to $\boldsymbol{\nabla}_\perp^2 A_{1\parallel}$ differ due to the differences identified in the expressions for the second order Hamiltonians H_2^{full} and H_2^{ORB5}. The same issue results in differences in terms proportional to the gradient of the background distribution function $\boldsymbol{\nabla} F_0$.

A slightly different version of (106) is implemented in the ORB5, the last term (only for ions contribution) is written down in the following form:

$$\int dW\; \boldsymbol{\nabla}\cdot\left(\frac{\mu}{B} F_0 \boldsymbol{\nabla}_\perp A_{1\parallel}\right), \tag{107}$$

which is more convenient for weak formulation of Ampère's equation suitable for numerical implementation into the PIC code. The final Ampère's law for ORB5 neglects the third term in the r.h.s. of (106):

$$\sum_{\mathrm{sp}} \frac{1}{d_{\mathrm{sp}}^2} A_{1\parallel} + \frac{1}{2}\sum_{\mathrm{sp}\neq \mathrm{e}} \nabla\cdot\left(\beta_{sp}\boldsymbol{\nabla}_\perp A_{1\parallel}\right) - \boldsymbol{\nabla}_\perp^2 A_{1\parallel} = \sum_{\mathrm{sp}} 4\pi \int dV\; dW\; \frac{e\, p_z}{mc}\mathcal{J}_0^{\mathrm{gc}\dagger}\; F_1 \tag{108}$$

where we have defined $d_{\mathrm{sp}} \equiv \frac{4\pi e^2 n_{0e}}{mc^2}$ and $\beta_{\mathrm{sp}} = \frac{8\pi\mu n_{\mathrm{sp}}}{B^2} \equiv \frac{8\pi n_{\mathrm{sp}} k_B T_{\mathrm{sp}}}{B^2}$.

1. Vlasov equation

Let us now compare Vlasov equations, we can derive from both Eulerian action functionals via variational calculation.

As we have seen, the full second order Eulerian action provides Vlasov equation with non-linear drive terms (86), while the Eulerian action functional containing physical reduction uses first order gyrocenter characteristics (31) to reconstruct Vlasov equation and does not contain the nonlinear drive term.

Finally, the following equation is solved in the ORB5:

$$\frac{dF}{dt} \equiv \{\mathcal{H}_1, \mathcal{F}\}_{\text{ext}} = 0 \tag{109}$$

which represents basically the same equation as (86), where we have taken into account that the background distribution is non-dynamical $\{F_0, H_0\}_{\text{gc}} = 0$ and static $\partial_t F_0 = 0$, which leads to the following:

$$\frac{dF_1}{dt} = -\{F_0, H_1\}_{\text{gc}} \tag{110}$$

In the other words, the dynamics of the dynamical part of the distribution function F_1 is defined from the linear evolution of the background distribution.

VI. ENERGY CONSERVATION DIAGNOSTICS

The energy diagnostics implemented in electromagnetic version of code ORB5 are derived from the energy conservation law, which corresponds to the electromagnetic Lagrangian (100).

Intuitively, the conserved energy density can be written down as follows:

$$\begin{aligned}\mathcal{E}^{\mathsf{ORB5}} &= \sum_{\text{sp}} \int dW\ dV\ H_0\ (F_0 + \epsilon_\delta F_1) + \sum_{\text{sp}} \int dW\ dV\ H_1\ (F_0 + \epsilon_\delta F_1) \\ &+ \sum_{\text{sp}} \int dW\ dV\ H_2^{\mathsf{ORB5}}\ F_0 + \int dV\ \frac{|\boldsymbol{\nabla}_\perp A_{1\parallel}|^2}{8\pi}\end{aligned} \tag{111}$$

which is equivalent to the result coming out from the direct Noether method application in the framework of the Eulerian variational principle, once we have taken into account the Poisson equation corresponding to the truncated Lagrangian. Below we give a detailed explanation.

First, we explicitly write the expression for the second term

$$\int dV\ dW\ H_1\ (F_0 + \epsilon_\delta F_1) = \int dV\ dW\ \ (F_0 + \epsilon_\delta F_1)\, \mathcal{J}_0^{\text{gc}}(\phi_1) \tag{112}$$

$$- \int dV\ dW\ \frac{ep_z}{mc}\ (F_0 + \epsilon_\delta F_1)\, \mathcal{J}_0^{\text{gc}}\left(A_{1\parallel}\right), \tag{113}$$

where we have used the explicit definition for the gyrocenter gyro-averaging operator (88) to expand the expression for H_1 as follows: $\langle \phi_{1gc} \rangle \equiv \langle \phi_1(\mathbf{X} + \boldsymbol{\rho}_0) \rangle = \langle \phi_1(\mathbf{r}) \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r}) \rangle \equiv \mathcal{J}_0^{\text{gc}}(\phi_1)$

Next, with using the quasi neutrality equation and integrating by parts:

$$\sum_{\text{sp}} \int dV\ dW\ F_0\ \frac{mc^2}{B^2} \left|\boldsymbol{\nabla}_\perp \phi_1\right|^2 \ = \ -\sum_{\text{sp}} \int dV\ dW \mathcal{J}_0^{\text{gc}\dagger} \left(F_0 + \epsilon_\delta F_1\right)\ \phi_1 \tag{114}$$

$$= \ -\sum_{\text{sp}} \int dV\ dW \left(F_0 + \epsilon_\delta F_1\right) \mathcal{J}_0^{\text{gc}} \left(\phi_1\right) \tag{115}$$

Finally, with taking into account the expression for the second order Hamiltonian H_2^{ORB5}, we get:

$$\mathcal{E}^{\mathsf{ORB5}} \ = \ \sum_{\text{sp}} \int dV\ dW\ \left(H_0 - \epsilon_\delta\ e\frac{p_z}{m}\ \mathcal{J}_0^{\text{gc}} \left(A_{1\|}\right)\right)\ \left(F_0 + \epsilon_\delta F_1\right) + \sum_{\text{sp}} \int dV \frac{\left|\boldsymbol{\nabla}_\perp A_{1\|}\right|^2}{8\pi} \tag{116}$$

$$+ \ \sum_{\text{sp}} \int dV\ dW \left(\frac{e^2}{2mc^2} A_{1\|}^2 + \frac{mc^2}{2B^2} \left|\nabla_\perp \phi_1\right|^2\right) F_0 \equiv \mathcal{E}_k + \mathcal{E}_F$$

which corresponds to the energy density obtained from the direct application of the Noether method in Eurlerian variational framework with truncated Hamiltonian corresponding to the ORB5 model.

We refer the first term, which contains only the unperturbed Hamiltonian H_0 as a kinetic energy $\mathcal{E}_k$ and the other terms as a field energy $\mathcal{E}_F$.

Let us now proceed with code diagnostics derivation. What can be measured in the PIC code in order to control the quality of the simulations?

It is well known that into the PIC code particles and fields are evaluated in two different ways. Particles are advanced along their characteristics without use of any grid, while fields are evaluated on the grid.

So, to control the quality of the simulation the contributions from particles energy and the field energy should be calculated independently.

This is why we are considering the power balance equation, called also the $E \times B$ transfer equation:

$$\frac{d\mathcal{E}_k}{dt} = -\frac{d\mathcal{E}_F}{dt} \tag{117}$$

The contributions from the particles dynamics are contained into the kinetic part $\mathcal{E}_k$ of the conserved energy density $\mathcal{E}$.

$$\frac{d\mathcal{E}_k}{dt}\left(t\right) = \sum_{\text{sp}}\ \int dW\ dV\ \frac{dH_0}{dt}\ \left(F_0 + \epsilon_\delta F_1\right) + \sum_{\text{sp}} \int dW\ dV\ H_0 \underbrace{\frac{dF}{dt}}_{\equiv 0} \tag{118}$$

The first term here represents explicit time derivative of the guiding-center Hamiltonian H_0. The last term here vanishes because of the Liouville theorem. And the remaining terms do not contain any dynamical fields.

To derive the diagnostics for the field part of the energy, as it is measured in the simulations, we need to use both: the corresponding quasineutrality and the Ampère equations, but this time to replace polarization terms related to the background distribution F_0 by the moments of the distribution function $F_0 + \epsilon_\delta F_1$.

We start with writing the quasi neutrality equation in a *weak* form, taking into account additional integration by parts in order to replace the guiding-center gyro averaged operator $\mathcal{J}_0^{\rm gc}$ from the electrostatic potential to the distribution function $(F_0 + \epsilon_\delta F_1)$

$$\sum_{\rm sp} \int dV dW \left(\frac{mc^2}{B^2} F_0 \right) |\boldsymbol{\nabla}_\perp \phi_1|^2 = \sum_{\rm sp} \int dV dW \mathcal{J}_0^{\rm gc\dagger} \left(F_0 + \epsilon_\delta F_1 \right) \ \phi_1 \tag{119}$$

Next, we reconstruct the Ampère's equation corresponding to the ORB5 code in a *weak* form by combining the field terms in the energy density expression (111) and with using again the integration by parts to change place of the gyroaveraged operator $\mathcal{J}_0$

$$- \int dV \ dW \ F_0 \ \frac{e^2}{2mc^2} A_{1\|}^2 + \int dV \ dW \ \frac{|\boldsymbol{\nabla}_\perp A_{1\|}|^2}{8\pi} = \frac{1}{2} \int dV \ dW \ \frac{e p_z}{mc} \mathcal{J}_0^{\rm gc\dagger} \ \left(F_0 + \epsilon_\delta F_1 \right) A_{1\|} \tag{120}$$

That operation leads to the following expression for the field energy term associated with the second order reduced dynamics, implemented into the energy balance equation (117) :

$$\mathcal{E}_F = \sum_{\rm sp} \frac{1}{2} \int dV \ dW \ \mathcal{J}_0^{\rm gc\dagger} \left(F_0 + \epsilon_\delta F_1 \right) \ \phi_1 - \sum_{\rm sp} \frac{1}{2} \int dV \ dW \ \mathcal{J}_0^{\rm gc\dagger} \left(F_0 + \epsilon_\delta F_1 \right) \ \frac{e p_z}{mc} \ A_{1\|}. \tag{121}$$

Therefore, the final expression for the energy density $\mathcal{E}$ can be rewritten as:

$$\mathcal{E} = \sum_{\rm sp} \int dV dW \left(F_0 + \epsilon_\delta F_1 \right) \left[\left(\frac{p_z^2}{2m} + \mu B \right) + \frac{e}{2} \mathcal{J}_0^{\rm gc} \left(\phi_1 \right) - \frac{p_z}{2mc} \mathcal{J}_0^{\rm gc} \left(A_{1\|} \right) \right] \tag{122}$$

We evaluate the time derivative of the kinetic energy $\mathcal{E}_k$ with using the first order gyrocenter characteristics for the phase space coordinates $\dot{p}_z$ and $\dot{\mathbf{X}}$:

$$\begin{aligned} \frac{d\mathcal{E}_k}{dt} (t) &= \sum_{\rm sp} \int dV dW \left(F_0 + \epsilon_\delta F_1 \right) \left[\frac{p_z}{m} \ \dot{p}_{z \ {\rm gy}}^{(1)} + \sum_{\rm sp} \mu \dot{\mathbf{X}}_{\rm gy}^{(1)} \cdot \boldsymbol{\nabla} B \right] \\ &= \sum_{\rm sp} \int dV dW \left(F_0 + \epsilon_\delta F_1 \right) \ \left[- e \boldsymbol{\nabla} \mathcal{J}_0^{\rm gc} \left(\psi_1 \right) \cdot \dot{\mathbf{X}} \Big|_0 + \frac{1}{c} \mathcal{J}_0^{\rm gc} \left(A_{1\|} \right) \left(\frac{\dot{p}_z}{m} \right) \Big|_0 \right] \end{aligned} \tag{123}$$

The details of that calculation can be found in the appendix B.

For practical reasons, in numerical simulations, it is particularly useful to consider the power balance equation in the following form (i.e. normalized by the field energy $\mathcal{E}_F$):

$$\frac{1}{\mathcal{E}_F}\frac{d\mathcal{E}_k}{dt} = -\frac{1}{\mathcal{E}_F}\frac{d\mathcal{E}_F}{dt} \tag{124}$$

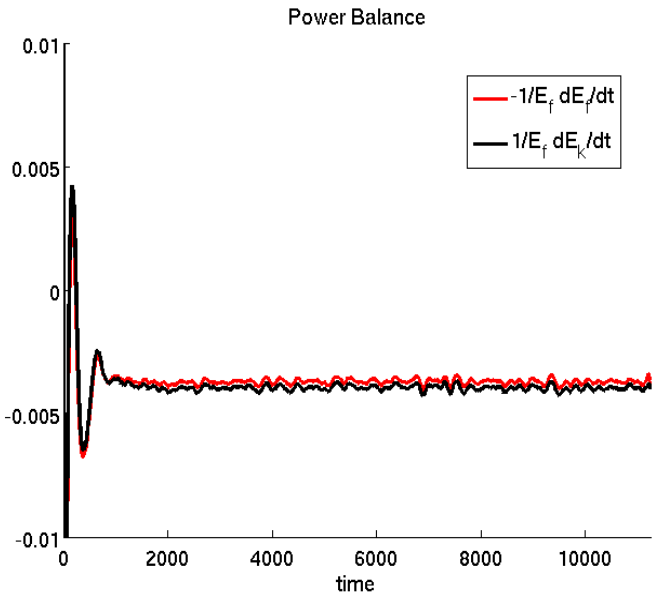

FIG. 4: Time evolution of the right-hand side and the left hand side of the power balance equation (124) for the linear CYCLONE base case simulations with ORB 5 code.

In linear simulations, the power balance equation (124) not only gives an indication about quality of the simulation but also can be used for measuring the instantaneous growth rate of the instability:

$$\gamma = \frac{1}{2}\frac{d}{dt}\log\mathcal{E}_F = \frac{1}{2}\frac{1}{\mathcal{E}_F}\frac{d\mathcal{E}_F}{dt}. \tag{125}$$

Hence, with taking into account (123):

$$\gamma = \frac{1}{2\mathcal{E}_F}\sum_{\text{sp}}\int dV\ dW\ \ (F_0+\epsilon_\delta F_1)\left[e\boldsymbol{\nabla}\left\langle\psi_{1gc}\right\rangle\cdot\dot{\mathbf{X}}\Big|_0 - \frac{1}{c}\left\langle A_{1\|gc}\right\rangle\left(\frac{\dot{p}_z}{m}\right)\Big|_0\right] \tag{126}$$

We present on the two following figures examples of the diagnostics implementation for a different type of instabilities: Electromagnetic ITG and the KBM .

The different contributions to the growth rate γ arising from the different terms in the unperturbed guiding-centre characteristics $\dot{\mathbf{X}}|_0$ and $\dot{p}_z|_0$ can be separated in the power balance equation and give a clear vision of which type of the instability is present in the system: that diagnostics is suitable for both linear and nonlinear simulations:

$$\gamma = \frac{1}{2\mathcal{E}_F}\sum_{\rm sp}\int dV\ dW\left(F_0+\epsilon_\delta F_1\right)\boldsymbol{\nabla}\mathcal{J}_0^{\rm gc}\left(\psi_1\right)\left(v_\parallel+v_{\boldsymbol{\nabla}P}+v_{\boldsymbol{\nabla}B}\right)$$
$$-\frac{1}{2\mathcal{E}_F}\sum_{\rm sp}\int dV\ dW\left(F_0+\epsilon_\delta F_1\right)\boldsymbol{\nabla}\mathcal{J}_0\left(A_{1\parallel}\right)\left(\mu B\boldsymbol{\nabla}\cdot\widehat{\mathbf{b}}+\frac{\mu c}{eB_\parallel^*}p_z\widehat{\mathbf{b}}\times\left(\widehat{\mathbf{b}}\times\frac{\boldsymbol{\nabla}\times\mathbf{B}}{B}\right)\cdot\boldsymbol{\nabla}B\right)$$

where

$$v_\parallel\equiv\frac{p_z}{m}\widehat{\mathbf{b}} \tag{127}$$

$$v_{\boldsymbol{\nabla}P}\equiv-\left(\frac{p_z}{m}\right)^2\frac{mc}{eB_\parallel^*}\widehat{\mathbf{b}}\times\frac{\boldsymbol{\nabla}P}{B^2} \tag{128}$$

$$v_{\boldsymbol{\nabla}B}\equiv\left(\frac{\mu B}{m}+\left(\frac{p_z}{m}\right)^2\right)\frac{mc}{eB_\parallel^*}\widehat{\mathbf{b}}\times\frac{\boldsymbol{\nabla}B}{B} \tag{129}$$

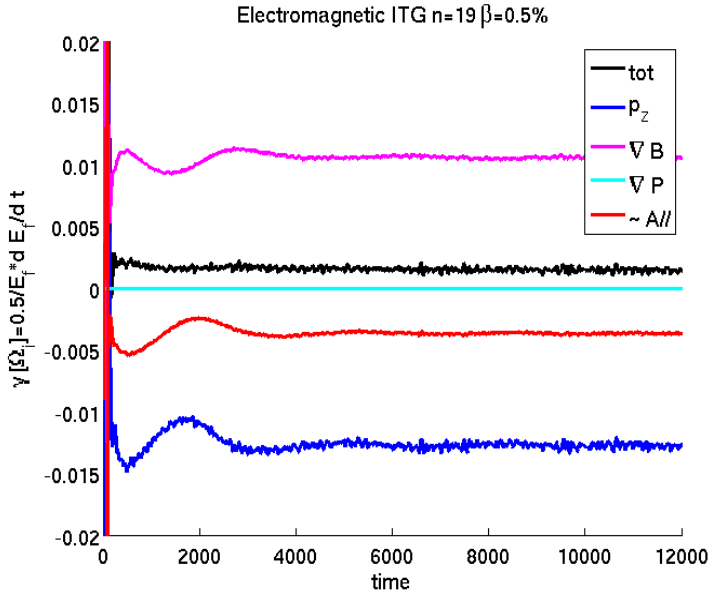

FIG. 5: The Ion Temperature Gradient instability: time evolution of different contributions to the instantaneous growth rate for the most unstable mode of the linear CYCLONE base case.

VII. CONCLUSIONS AND PERSPECTIVES

In this work we have presented two variational derivations of self-consistent gyrokinetically reduced systems of Maxwell-Vlasov equations containing the second order corrections with respect

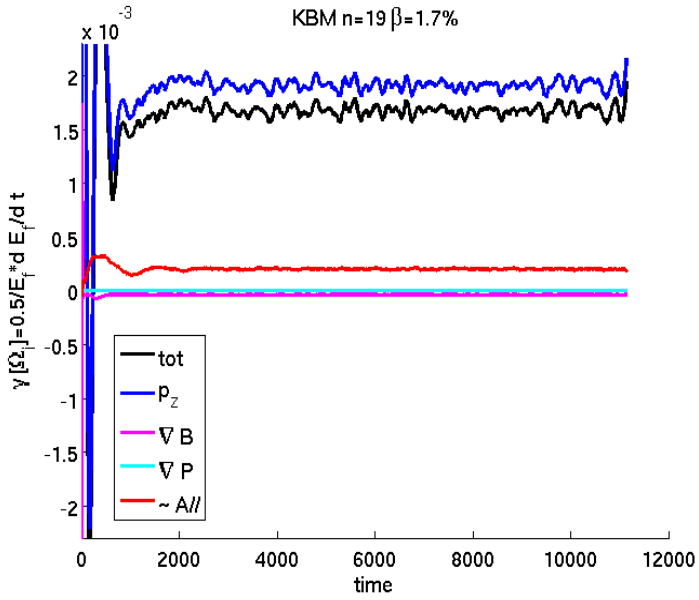

FIG. 6: The Kinetic Balooning Mode instability: time evolution of different contributions to the instantaneous growth rate for the most unstable mode of the linear CYCLONE base case.

to the small parameter related to the gyrocenter dynamical reduction ϵ_δ and up to the second order with respect to the FLR corrections. The first system issued from the direct derivation uses second order truncated Eulerian variational action functional for gyrokinetic Maxwell-Vlasov system. The second system of the reduced gyrokinetic Maxwell-Vlasov equations uses physical approximation and derives from the Lagrangian variational principle, suitable for PIC codes implementations.

The main results of this work can be summarized in the following way. The electrostatic limit of both models coincides. The electrostatic model, implemented into the ORB5 is energetically consistent. In what concerns the electromagnetic case, here several differences exist. First of all, due to the differences in choice of the second order Hamiltonian (direct second order gyrokinetic dynamical reduction for particles or the physical model), the gyrokinetic equations for fields may or may not be coupled. Also terms proportional to $\nabla_\perp^2 A_{1\|}$ have different expressions.

Neglecting magnetic field curvature into the model issued from the physical approximation violates the Maxwell constraint $\nabla \cdot \mathbf{B} = 0$, which can be a potential issue while implementing a phase space preserving numerical scheme.

Finally, the reduced Vlasov equations coincide up to the non-linear term $\{F_1, H_1\}_{\mathrm{gc}}$, issued from the direct second order derivation, which is absent in a physical model.

At the next step of our work we aim to proceed with derivation of the weak form for (90) and

(92) suitable for the finite element method discretization necessary for further implementation of new coupled system into the ORB5.

In the same time, further comparison of the established second order reference model (90, 92) with gyrokinetic equations implemented into other codes involved into the Enabling Research project is one of our highest priorities.

VIII. ACKNOWLEDGMENTS

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Appendix A: Explicit derivation of the full second order Maxwell-Vlasov equations

We use the same organisation of the action functional (80) into the field and Vlasov parts as given by (44) in the section (III A), while deriving equations of motion and conservation laws into the implicit form. In addition to that we separate the nonlinear Vlasov part in two terms:

$$\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{nonlin}}[\phi_1, A_{1\parallel}] \equiv \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}\left[\phi_1, A_{1\parallel}\right] + \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}}\left[\phi_1, A_{1\parallel}\right]. \tag{A1}$$

Those two nonlinear Vlasov terms are associated to the second order Hamiltonian H_2 truncated at the second order of the FLR decomposition. and contain two different polarization contributions: the electromagnetic part $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}\left[\phi_1, A_{1\parallel}\right]$, given by (83) and a purely magnetic contribution with the first and the second FLR corrections $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}}\left[\phi_1, A_{1\parallel}\right]$, given by (84).

The first variation of the Eulerian action functional contains three parts, each one corresponds to the functional dependence in variational fields $(\phi_1, A_{1\parallel}, \mathcal{F})$:

$$\delta\mathcal{A}^{\mathcal{E}}\left[\phi_1, A_{1\parallel}, \mathcal{F}\right] = \frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\phi_1} \circ \widehat{\phi}_1 + \frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} + \frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\mathcal{F}} \circ \widehat{\mathcal{F}}, \tag{A2}$$

where $\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\phi_1}$ is the functional derivative with respect to the electrostatic potential ϕ_1, $\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta A_{1\parallel}}$ with respect to the parallel magnetic potential perturbation $A_{1\parallel}$, and finally $\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\mathcal{F}}$ contains functional

derivatives with respect to the extended Vlasov field, given by (36). We remind that all of those objects are linear functionals and need to be implied to some test function.

a. Fields contributions. Parallel magnetic perturbation

Taking into account the expression for perturbative magnetic field (76) we calculate functional derivatives with respect to the parallel component of the magnetic potential $A_{1\parallel}$; we choose the test function $\widehat{\mathbf{A}}_1 \equiv \widehat{A}_{1\parallel}\widehat{\mathbf{b}}$.

Following the implicit derivation presented into the section (III A), we start by calculating functional derivative with respect to $A_{1\parallel}$(50).

Thus, the expression for the dynamical and the Noether's terms of the $A_{1\parallel}$, the functional derivative of Maxwell's part of the action functional $\mathcal{A}^{\mathcal{E}}$:

$$\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{magn}}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} = -\epsilon_\delta \int \frac{dV}{4\pi} (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \boldsymbol{\nabla} \cdot \left(\widehat{A}_{1\parallel} \widehat{\mathbf{b}} \right) \tag{A3}$$

$$= \epsilon_\delta \int \frac{dV}{4\pi} \, \widehat{A}_{1\parallel} \, \widehat{\mathbf{b}} \cdot (\boldsymbol{\nabla} \times [\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1]) \tag{A4}$$

$$+ \epsilon_\delta^2 \int \frac{dV}{4\pi} \boldsymbol{\nabla} \cdot \left(\widehat{A}_{1\parallel} \left[\boldsymbol{\nabla}_\perp A_{1\parallel} + A_{1\parallel} \, \widehat{\mathbf{b}} \times \boldsymbol{\nabla} \times \widehat{\mathbf{b}} \right] \right).$$

As we can see, the dynamical term is now multiplied by the unitary vector of the background magnetic field $\widehat{\mathbf{b}}$, it means that the resulting Ampere's equation will be projected on the parallel direction. This is a direct consequence of the perturbed magnetic field choice (76).

For the Noether's term we have used the following decomposition:

$$(\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \widehat{\mathbf{A}}_{1\parallel} \equiv (\mathbf{B}_0 + \epsilon_\delta \mathbf{B}_1) \times \left(\widehat{\mathbf{b}} \, \widehat{A}_{1\parallel} \right)$$
$$= -\widehat{A}_{1\parallel} \left(\boldsymbol{\nabla} A_{1\parallel} \times \widehat{\mathbf{b}} \right) \times \widehat{\mathbf{b}} \; - \widehat{A}_{1\parallel} A_{1\parallel} \left(\boldsymbol{\nabla} \times \widehat{\mathbf{b}} \right) \times \widehat{\mathbf{b}}.$$

In order to get explicit expressions for the reduced Maxwell-Vlasov equations of motion we need to explicitly evaluate contributions from H_1 and H_2, chosen to represent reduced particle dynamics into the Eulerian action.

b. First order Vlasov contributions

The direct calculation from the second order Eulerian action gives us the following form of the first order Vlasov contributions:

$$\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta \phi_1} \circ \widehat{\phi}_1 = -\epsilon_\delta \int dV\ dW\ \ (F_0 + \epsilon_\delta F_1)\ \left\langle \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\ \widehat{\phi}_1(\mathbf{r}) \right\rangle \tag{A5}$$

$$\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} = \epsilon_\delta \int dV\ dW\ \ (F_0 + \epsilon_\delta F_1)\ \frac{p_z}{mc} \left\langle \delta(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r})\ \widehat{A}_{1\parallel}(\mathbf{r}) \right\rangle, \tag{A6}$$

where we have used:

$$\begin{aligned} \frac{\delta H_1}{\delta \phi_1} \circ \widehat{\phi}_1 \ &= \ \frac{\delta \left\langle \phi_{1\mathrm{gc}} \right\rangle}{\delta \phi_1} \circ \widehat{\phi}_1(\mathbf{r}) = \frac{\delta \left\langle \phi_1 \left(\mathbf{X} + \boldsymbol{\rho}_0 \right) \right\rangle}{\delta \phi_1(\mathbf{r})} \circ \widehat{\phi}_1(\mathbf{r}) \qquad \text{(A7)} \\ &= \ \left\langle \left(\delta^3(\mathbf{X} + \boldsymbol{\rho}_0 - \mathbf{r}) \widehat{\phi}_1(\mathbf{r}) \right) \right\rangle = \left\langle \left(\widehat{\phi}_1(\mathbf{X} + \boldsymbol{\rho}_0) \right) \right\rangle \equiv \mathcal{J}^{\mathrm{gc}}_0 \left(\phi_1 \right) \\ \frac{\delta H_1}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} &= \frac{p_z}{mc} \frac{\delta \left\langle \left(A_{1\parallel gc} \right) \right\rangle}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} = \frac{p_z}{mc} \left\langle \left(\widehat{A}_{1\parallel} \left(\mathbf{X} + \boldsymbol{\rho}_0 \right) \right) \right\rangle \equiv \frac{p_z}{mc} \mathcal{J}^{\mathrm{gc}}_0 \left(A_{1\parallel} \right) \qquad \text{(A8)} \end{aligned}$$

c. Second order Vlasov contributions

Let us now explicitly calculate contributions from the nonlinear Vlasov terms associated with the second order Hamiltonian H_2. We separate the latter in two parts: $H_{2\mathsf{polmix}}$ and $H_{2\mathsf{polmag}}$, which will generate $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}$ and $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}}$ nonlinear Vlasov contributions to the Eulerian action defined in (A1). Splitting the expression (79) of H_2 we get:

$$H_{2\mathsf{polmix}} = -\frac{mc^2}{2B^2} \left| \boldsymbol{\nabla}_\perp \phi_1 - \frac{p_z}{mc} \boldsymbol{\nabla}_\perp A_{1\parallel} \right|^2 \tag{A9}$$

and

$$H_{2\mathsf{polmag}} = \frac{e^2}{2mc^2} A^2_{1\parallel} + \frac{\mu}{2B} \left| \boldsymbol{\nabla}_\perp A_{1\parallel} \right|^2 + \frac{1}{2} \frac{\mu}{B} A_{1\parallel} \boldsymbol{\nabla}^2_\perp A_{1\parallel} \tag{A10}$$

The $H_{2\mathsf{polmix}}$ part of Hamiltonian will contribute into the $\delta\phi_1$ and $\delta A_{1\parallel}$ parts of the first derivative of H_2, while purely magnetic part $H_{2\mathsf{polmag}}$ will naturally provide only $\delta A_{1\parallel}$ associated contribution:

$$\delta H_2 = \frac{\delta H_{2\mathsf{polmix}}}{\delta \phi_1} \circ \widehat{\phi}_1 + \left(\frac{\delta H_{2\mathsf{polmix}}}{\delta A_{1\parallel}} + \frac{\delta H_{2\mathsf{polmag}}}{\delta A_{1\parallel}} \right) \circ \widehat{A}_{1\parallel} \tag{A11}$$

Let us start with the contributions from $H_{2\mathsf{polmix}}$ given by (A9) contributions:

$$-\int dV\ dW\ F_0\ \delta H_{2\mathsf{polmix}} \equiv \frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta \phi_1} \circ \widehat{\phi}_1 + \frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} \tag{A12}$$

We compute the first variation using the functional derivative definition (38):

$$
\begin{aligned}
&-\int dV\ dW\ F_0\ \delta H_{2\mathsf{polmix}} = \\
&= \epsilon_\delta^2\ \frac{d}{d\nu}\left[\int dV\ dW\ \frac{mc^2}{2B^2}\ F_0\left|\left(\boldsymbol{\nabla}_\perp\phi_1+\nu\ \boldsymbol{\nabla}_\perp\widehat{\phi}_1\right)-\frac{p_z}{mc}\left(\boldsymbol{\nabla}_\perp A_{1\parallel}+\nu\ \boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel}\right)\right|^2\right]\Bigg|_{\nu=0} \\
&= \epsilon_\delta^2\left[\int dV\ dW\ \frac{mc^2}{B^2}\ F_0\ \boldsymbol{\nabla}_\perp\left(\phi_1-\frac{p_z}{mc}A_{1\parallel}\right)\cdot\boldsymbol{\nabla}_\perp\left(\widehat{\phi}_1-\frac{p_z}{mc}\widehat{A}_{1\parallel}\right)\right]
\end{aligned}
$$

This is the *weak* form of the Vlasov contribution from (A9).

In order to separate dynamical and Noether's contributions, we integrate that expression by parts. Let us analyse in details the electrostatic term with test function $\widehat{\phi}_1$, then the parallel magnetic potential contribution with $\widehat{A}_{1\parallel}$ could be obtained in a similar way. First, we remind that the phase space volume $dV\ dW \equiv d^3\mathbf{X}\ B_\parallel^*\ dp_z\ d\mu$ appearing in integrals contains guiding-center Jacobian $B_\parallel^* = B_\parallel^*(\mathbf{X}, p_z, \mu)$, so we need to pay attention on that phase-space functional dependence while using Leibnitz rule.

$$
\begin{aligned}
&\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta\phi_1}\circ\widehat{\phi}_1 = \int B_\parallel^*\ dV\ dp_z\ d\mu\ \left(\frac{mc^2}{B^2}\ F_0\right)\left[\boldsymbol{\nabla}_\perp\phi_1-\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right]\cdot\boldsymbol{\nabla}_\perp\widehat{\phi}_1 \\
&= \int dV\ dp_z\ d\mu\ \boldsymbol{\nabla}_\perp\left[B_\parallel^*\ \left(\frac{mc^2}{B^2}F_0\right)\left(\boldsymbol{\nabla}_\perp\phi_1-\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right)\widehat{\phi}_1\right] \qquad \text{(A13)} \\
&- \int dV\ \boldsymbol{\nabla}_\perp\left[\int\ B_\parallel^*\ dp_\parallel\ d\mu\left(\frac{mc^2}{B^2}F_0\right)\left(\boldsymbol{\nabla}_\perp\phi_1-\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right)\right]\widehat{\phi}_1 \qquad \text{(A14)}
\end{aligned}
$$

Here (A13) is a Noether contribution and (A14) is a dynamical part.

Similarly, for the parallel magnetic potential perturbation contribution, we have:

$$
\begin{aligned}
&\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta A_{1\parallel}}\circ\widehat{A}_{\parallel 1} = \int B_\parallel^*\ dV\ dp_z\ d\mu\ \left(\frac{mc^2}{B^2}\ F_0\right)\frac{p_z}{mc}\left[-\boldsymbol{\nabla}_\perp\phi_1+\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right]\boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel} \\
&= \int dV\ dp_z\ d\mu\ \boldsymbol{\nabla}_\perp\left[B_\parallel^*\ \left(\frac{mc^2}{B^2}F_0\right)\frac{p_z}{mc}\left(-\boldsymbol{\nabla}_\perp\phi_1+\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right)\widehat{A}_{1\parallel}\right] \qquad \text{(A15)} \\
&- \int dV\ \boldsymbol{\nabla}_\perp\left[\int\ B_\parallel^*\ dp_z\ d\mu\left(\frac{mc^2}{B^2}F_0\right)\frac{p_z}{mc}\left(-\boldsymbol{\nabla}_\perp\phi_1+\frac{p_z}{mc}\boldsymbol{\nabla}_\perp A_{1\parallel}\right)\right]\widehat{A}_{1\parallel} \qquad \text{(A16)}
\end{aligned}
$$

With introducing the equilibrium fluid density n_0 and the equilibrium current $\mathfrak{J}_0$:

$$
n_0 \equiv \int B_\parallel^* dp_z\ d\mu\ F_0 \qquad \text{(A17)}
$$

$$
\mathfrak{J}_0 \equiv c\ \int B_\parallel^* dp_z\ d\mu\ \frac{p_z}{mc}\ F_0, \qquad \text{(A18)}
$$

we can write:

$$\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta \phi_1} \circ \widehat{\phi}_1 \;=\; \int d^3\mathbf{X}\; \boldsymbol{\nabla}_\perp \left[\frac{mc^2}{B^2} \left(n_0 \boldsymbol{\nabla}_\perp \phi_1 - \frac{1}{c} \mathfrak{J}_0 \boldsymbol{\nabla}_\perp A_{1\parallel} \right)\; \widehat{\phi}_1 \right] \tag{A19}$$

$$- \int d^3\mathbf{X}\; \boldsymbol{\nabla}_\perp \left[\frac{mc^2}{B^2} \left(n_0 \boldsymbol{\nabla}_\perp \phi_1 - \frac{1}{c} \mathfrak{J}_0 \boldsymbol{\nabla}_\perp A_{1\parallel} \right) \right] \widehat{\phi}_1 \tag{A20}$$

Then by proceeding with the similar calculation, and defining the moment of the equilibrium distribution function weighted by $\left(\frac{p_z}{mc}\right)^2$ as

$$\widetilde{\mathfrak{J}}_0 \equiv c^2 \int B^*_\parallel dp_z\; d\mu\; \frac{p_z}{mc}\; F_0 \tag{A21}$$

we can write the electromagnetic part as:

$$\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} \;=\; \int d^3\mathbf{X}\; \boldsymbol{\nabla}_\perp \left[\frac{mc^2}{B^2} \left(-\frac{1}{c} \mathfrak{J}_0 \boldsymbol{\nabla}_\perp \phi_1 + \frac{1}{c^2} \widetilde{\mathfrak{J}}_0 \boldsymbol{\nabla}_\perp A_{1\parallel} \right)\; \widehat{A}_{1\parallel} \right] \tag{A22}$$

$$- \int d^3\mathbf{X}\; \boldsymbol{\nabla}_\perp \left[\frac{mc^2}{B^2} \left(-\frac{1}{c} \mathfrak{J}_0 \boldsymbol{\nabla}_\perp \phi_1 + \frac{1}{c^2} \widetilde{\mathfrak{J}}_0 \boldsymbol{\nabla}_\perp A_{1\parallel} \right) \right] \widehat{A}_{1\parallel} \tag{A23}$$

Let us now analyse dynamical and Noether contributions coming from purely magnetic Vlasov part of the action, corresponding to the second order Hamiltonian $H_{2\mathsf{polmag}}$, which contains first and second order FLR corrections, given by first two terms in (79) and the third term correspondingly:

$$- \int dV\; dW\; F_0\; \delta H_{2\mathsf{polmag}} \equiv \frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}(1)}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} + \frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}(2)}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel}, \tag{A24}$$

where $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}(1)}$ contains the first order FLR correction from $H_{2\mathsf{polmag}}$ and $\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}(2)}$, the second order one.

Therefore, the contribution from the variation of the first order FLR term:

$$\begin{aligned}
\frac{\delta \mathcal{A}^{\mathcal{E}}_{\mathsf{polmag}\ (1)}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} \;&=\; -\frac{d}{d\nu}\; \frac{\epsilon_\delta^2}{2} \left\{ \int\; dV\; dW\; F_0 \left[\frac{1}{m} \frac{e^2}{c^2} \left(A_{1\parallel} + \nu\; \widehat{A}_{1\parallel} \right)^2 + \frac{\mu}{B} \left| \boldsymbol{\nabla}_\perp A_{1\parallel} + \nu \boldsymbol{\nabla}_\perp \widehat{A}_{1\parallel} \right|^2 \right] \right\}\Big|_{\nu=0}. \\
&=\; -\epsilon_\delta^2\; \frac{e^2}{mc^2} \int dV\; dW\; F_0\; A_{1\parallel}\; \widehat{A}_{1\parallel} - \epsilon_\delta^2 \int dV\; dW\; \frac{\mu}{B}\; F_0\; \boldsymbol{\nabla}_\perp A_{1\parallel} \cdot \boldsymbol{\nabla}_\perp \widehat{A}_{1\parallel} \qquad \text{(A25)} \\
&=\; -\epsilon_\delta^2\; \frac{e^2}{mc^2} \int dV\; dW\; F_0\; A_{1\parallel}\; \widehat{A}_{1\parallel} + \epsilon_\delta^2 \int dV d\mu\; dp_z \boldsymbol{\nabla}_\perp \left[B^*_\parallel\; F_0 \frac{\mu}{B} \boldsymbol{\nabla}_\perp A_{1\parallel} \right] \widehat{A}_{1\parallel} \\
&-\; \epsilon_\delta^2 \int dV\; d\mu\; dp_z \boldsymbol{\nabla}_\perp \left[B^*_\parallel\; F_0 \frac{\mu}{B} \boldsymbol{\nabla}_\perp A_{1\parallel}\; \widehat{A}_{1\parallel} \right]
\end{aligned}$$

The second contribution, we should account here is provided via the second order FLR term:

$$\begin{aligned}
\frac{\delta \mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}\ (2)}}{\delta A_{1\parallel}} \circ \widehat{A}_{1\parallel} \;&=\; -\frac{\epsilon_\delta^2}{2}\; \frac{d}{d\nu} \left\{ \int\; dV\; dW\; F_0\; \frac{\mu}{B} \left(A_{1\parallel} + \nu \widehat{A}_{1\parallel} \right) \boldsymbol{\nabla}_\perp^2 \left(A_{1\parallel} + \nu \widehat{A}_{1\parallel} \right) \right\}\Big|_{\nu=0} \\
&=\; -\frac{\epsilon_\delta^2}{2} \left\{ \int\; dV\; dW\; \left(F_0 \frac{\mu}{B} \right) \left(A_{1\parallel} \boldsymbol{\nabla}_\perp^2 \widehat{A}_{1\parallel} + \boldsymbol{\nabla}_\perp^2 A_{1\parallel}\; \widehat{A}_{1\parallel} \right) \right\} \qquad \text{(A26)}
\end{aligned}$$

The second term gives directly a dynamical contribution to the Ampere's equation. With using the Leibnitz rule two times, we rewrite the first term in order to obtain the Noether (exact derivative) and dynamical contributions.

$$-\frac{\epsilon_\delta^2}{2}\int dV\ dW\ \left(F_0\frac{\mu}{B}\right)\ \left(A_{1\parallel}\boldsymbol{\nabla}_\perp^2\widehat{A}_{1\parallel}\right) = \tag{A27}$$

$$-\ \frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\boldsymbol{\nabla}_\perp\left[B_\parallel^*\left(F_0\frac{\mu}{B}\right)\ A_{1\parallel}\boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel}\right]-\boldsymbol{\nabla}_\perp\left[B_\parallel^*F_0\frac{\mu}{B}\ A_{1\parallel}\right]\boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel}\right\} \tag{A28}$$

$$\begin{aligned}=\ &-\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\boldsymbol{\nabla}_\perp\left[\left(B_\parallel^*F_0\frac{\mu}{B}\right)\ A_{1\parallel}\boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel}\right]\right\}\\ &+\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\boldsymbol{\nabla}_\perp\left[\boldsymbol{\nabla}_\perp\left(B_\parallel^*F_0\frac{\mu}{B}\ A_{1\parallel}\right)\widehat{A}_{1\parallel}\right]\right\}-\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\boldsymbol{\nabla}_\perp^2\left[B_\parallel^*F_0\frac{\mu}{B}\ A_{1\parallel}\right]\widehat{A}_{1\parallel}\right\}\end{aligned}$$

Therefore for the second order FLR contribution we have:

$$\begin{aligned}\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{VI})}_{\mathsf{polmag}\ (2)}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel} = &-\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\ \boldsymbol{\nabla}_\perp\left[\left(B_\parallel^*F_0\frac{\mu}{B}\right)\ A_{1\parallel}\boldsymbol{\nabla}_\perp\widehat{A}_{1\parallel}\right]\right\}\\ &+\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\ \boldsymbol{\nabla}_\perp\left[\boldsymbol{\nabla}_\perp\left(B_\parallel^*F_0\frac{\mu}{B}\ A_{1\parallel}\right)\widehat{A}_{1\parallel}\right]\right\}\\ &-\frac{\epsilon_\delta^2}{2}\left\{\int dV\ d\mu\ dp_z\ \left[\boldsymbol{\nabla}_\perp^2\left[B_\parallel^*F_0\frac{\mu}{B}\ A_{1\parallel}\right]+\left(B_\parallel^*F_0\ \frac{\mu}{B}\right)\ \boldsymbol{\nabla}_\perp^2A_{1\parallel}\right]\widehat{A}_{1\parallel}\right\}\end{aligned} \tag{A29}$$

As we can see, there are two Noether contributions and two dynamical ones.

The first Noether term can be reorganized with using the Leibnitz rule as follows:

$$-\frac{\epsilon_\delta^2}{2}\nabla_\perp\left[\left(B_\parallel^*F_0\frac{\mu}{B}\right)A_{1\parallel}\nabla_\perp\widehat{A}_{1\parallel}\right]=-\frac{\epsilon_\delta^2}{2}\nabla_\perp^2\left[B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\widehat{A}_{1\parallel}\right]+\frac{\epsilon_\delta^2}{2}\nabla_\perp\left[\nabla_\perp\left(B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\right)\widehat{A}_{1\parallel}\right]$$

Then we write the contributions from the two Noether's terms together:

$$\begin{aligned}&-\frac{\epsilon_\delta^2}{2}\int dV\ d\mu\ dp_\parallel\ \nabla_\perp^2\left[B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\widehat{A}_{1\parallel}\right]+\epsilon_\delta^2\int dV\ d\mu\ dp_\parallel\ \nabla_\perp\left[\nabla_\perp\left(B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\right)\widehat{A}_{1\parallel}\right]\\ =\ &\frac{\epsilon_\delta^2}{2}\int dV\ d\mu\ dp_\parallel\ \nabla_\perp\left[2\ \nabla_\perp\left(B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\right)\widehat{A}_{1\parallel}-\nabla_\perp\left(B_\parallel^*F_0\frac{\mu}{B}A_{1\parallel}\widehat{A}_{1\parallel}\right)\right]\end{aligned} \tag{A30}$$

Next, with using the Leibnitz rule:

$$\nabla_\perp^2\left(B_\parallel^*F_0\frac{\mu}{B}A_\parallel\right)=\nabla_\perp^2\left(B_\parallel^*F_0\frac{\mu}{B}\right)A_\parallel+\left(B_\parallel^*F_0\frac{\mu}{B}\right)\nabla_\perp^2A_\parallel \tag{A31}$$

The dynamical contributions can be rewritten as:

$$-\frac{\epsilon_\delta^2}{2}\ \int dV\ d\mu\ dp_\parallel\left[\nabla_\perp^2\left(B_\parallel^*F_0\frac{\mu}{B}\right)A_{1\parallel}+2\ \left(B_\parallel^*F_0\frac{\mu}{B}\right)\nabla_\perp^2A_{1\parallel}\right]\ \widehat{A}_{1\parallel} \tag{A32}$$

As we can remark, the second dynamical term of the $\mathcal{A}^{\mathcal{E}(\mathsf{VI})}_{\mathsf{polmag}\ (2)}$ partially cancel dynamical contribution from $\mathcal{A}^{\mathcal{E}(\mathsf{VI})}_{\mathsf{polmag}\ (1)}$ so that no terms proportional to $\boldsymbol{\nabla}_\perp^2A_{1\parallel}$ will appear into the final Ampere's

law. Finally, contribution from the magnetic part of polarization can be written as:

$$\begin{aligned}\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}}}{\widehat{A}_{1\parallel}}\circ\widehat{A}_{1\parallel} &= -\epsilon_\delta^2\frac{e^2}{mc^2}\int dV\ dW\ F_0\ A_{1\parallel}\widehat{A}_{1\parallel}+\epsilon_\delta^2\int\ dV\ d\mu\ dp_\parallel\ \nabla_\perp\left(B^*_\parallel F_0\frac{\mu}{B}\right)\ \nabla_\perp A_{1\parallel}\widehat{A}_{1\parallel}\\ &-\ \frac{\epsilon_\delta^2}{2}\int\ dV\ d\mu\ dp_\parallel\ \nabla^2_\perp\left(B^*_\parallel F_0\frac{\mu}{B}\right)A_{1\parallel}\ \widehat{A}_{1\parallel}\\ &+\ \epsilon_\delta^2\int dV\ d\mu\ dp_\parallel\ \nabla_\perp\left[\nabla_\perp\left(B^*_\parallel F_0\frac{\mu}{B}A_{1\parallel}\right)\widehat{A}_{1\parallel}-\left(B^*_\parallel F_0\frac{\mu}{B}\right)\nabla_\perp A_{1\parallel}\widehat{A}_{1\parallel}\right]\\ &-\ \frac{\epsilon_\delta^2}{2}\int dV\ d\mu\ dp_\parallel\ \nabla^2_\perp\left[B^*_\parallel F_0\frac{\mu}{B}A_{1\parallel}\widehat{A}_{1\parallel}\right]\end{aligned}\tag{A33}$$

The Gyrokinetic Poisson equation is obtained by collecting dynamical contributions from different parts of functional derivative of Eulerian action with respect to the electrostatic potential ϕ_1:

$$0=\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta\phi_1}\circ\widehat{\phi}_1=\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}}{\delta\phi_1}\circ\widehat{\phi}_1+\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta\phi_1}\circ\widehat{\phi}_1+\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta\phi_1}\circ\widehat{\phi}_1,\tag{A34}$$

Comparing to the Poisson equation, the gyrokinetic Ampere equation has an additional contribution from the pure magnetic polarization term:

$$\frac{\delta\mathcal{A}^{\mathcal{E}}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel}=\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{field})}_{\mathsf{el}}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel}+\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmix}}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel}+\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{polmag}}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel}+\frac{\delta\mathcal{A}^{\mathcal{E},(\mathsf{Vl})}_{\mathsf{lin}}}{\delta A_{1\parallel}}\circ\widehat{A}_{1\parallel}\tag{A35}$$

Appendix B: Hamiltonian first order characteristics and ORB5 code diagnostics

In that section we give a detailed derivation of the first order gyrocenter characteristics in Hamiltonian representation. That will allow us to explicit the diagnostics implemented into the ORB5 code for control of the quality of the simulations.

$$\dot{\mathbf{X}}^{(1)}_{\mathrm{gy}}\ =\ \left\{\mathbf{X},\mathcal{H}^{(1)}_{\mathrm{gy}}\right\}_{\mathrm{ext}}=\frac{\mathbf{B}^*}{B^*_\parallel}\frac{\partial\mathcal{H}^{(1)}_{\mathrm{gy}}}{\partial p_z}+\frac{c\hat{\mathbf{b}}}{eB^*_\parallel}\times\nabla\mathcal{H}^{(1)}_{\mathrm{gy}}\tag{B1}$$

$$\dot{p}^{(1)}_{\mathrm{z\ gy}}\ =\ \left\{p_z,\mathcal{H}^{(1)}_{\mathrm{gy}}\right\}_{\mathrm{ext}}=-\frac{\mathbf{B}^*}{B^*_\parallel}\cdot\nabla\mathcal{H}^{(1)}_{\mathrm{gy}}.\tag{B2}$$

The symplectic magnetic field $\mathbf{B}^*$ writes:

$$\mathbf{B}^*=\mathbf{B}+\frac{e}{c}\ p_z\ \boldsymbol{\nabla}\times\widehat{\mathbf{b}}\tag{B3}$$

The geometric contribution to this symplectic field $\boldsymbol{\nabla}\times\widehat{\mathbf{b}}$ is expressed with using the projection on the parallel and perpendicular to the magnetic field directions:

$$\boldsymbol{\nabla}\times\widehat{\mathbf{b}}=\widehat{\mathbf{b}}\left(\widehat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\widehat{\mathbf{b}}\right)-\widehat{\mathbf{b}}\times\left[\widehat{\mathbf{b}}\times\boldsymbol{\nabla}\times\widehat{\mathbf{b}}\right]\equiv\widehat{\mathbf{b}}\ \tau-\mathbf{G},\tag{B4}$$

where the scalar τ represents the magnetic twist and the vector $\mathbf{G}$ is referred as magnetic curvature. Since $\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = -\nabla p$ in single fluid MHD equilibrium, we rewrite in the following way the curvature vector $\mathbf{G}$ in order to make explicitly appear the pressure-like contributions into the characteristics.

$$\mathbf{G} = \widehat{\mathbf{b}} \times \left[\widehat{\mathbf{b}} \times \frac{\boldsymbol{\nabla} \times \mathbf{B}}{B} \right] + \frac{\boldsymbol{\nabla} B \times \widehat{\mathbf{b}}}{B} \tag{B5}$$

Therefore, we can also decompose the symplectic magnetic field into the parallel and the perpendicular components in a following way:

$$\mathbf{B}^* = \underbrace{\left(B + \frac{c}{e} p_z \, \tau \right)}_{\equiv B_\parallel^*} \widehat{\mathbf{b}} - \frac{c}{e} p_z \mathbf{G} \tag{B6}$$

The final expressions for characteristics are implemented into the code in a following form:

$$\dot{\mathbf{X}}^{(1)}_{\text{gy}} = \frac{p_z}{m} \widehat{\mathbf{b}} - \left(\frac{p_z}{m} \right)^2 \frac{m}{e B_\parallel^*} \widehat{\mathbf{b}} \times \left(\widehat{\mathbf{b}} \times \frac{\boldsymbol{\nabla} \times \mathbf{B}}{B} \right) + \left(\frac{\mu}{m} + \left(\frac{p_z}{m} \right)^2 \right) \frac{m}{e B_\parallel^*} \widehat{\mathbf{b}} \times \frac{\boldsymbol{\nabla} B}{B} \tag{B7}$$

$$- \frac{e}{c} \left\langle A_{1\parallel gc} \right\rangle \widehat{\mathbf{b}} + \frac{p_z}{m} \frac{1}{B_\parallel^*} \left\langle A_{1\parallel gc} \right\rangle \mathbf{G} - \frac{1}{B_\parallel^*} \boldsymbol{\nabla} \left\langle \psi_{1\text{gc}} \right\rangle \times \widehat{\mathbf{b}} \tag{B8}$$

$$\equiv \text{vpar} + \text{vpressure} + \text{vgradb} + \text{vapar1} + \text{vexb} \tag{B9}$$

The first three terms represent non-perturbed (guiding-center) characteristics with vpar the parallel velocity, vpressure the pressure-like term and vgradb containing the gradient of magnetic field amplitude $\boldsymbol{\nabla} B$. Two next terms contains the perturbed gyrocenter electromagnetic potential $\left\langle A_{1\parallel gc} \right\rangle$ and are referred as vapar1; the last term vexb is electromagnetic $E \times B$- velocity.

The characteristic for p_z coordinate:

$$\dot{p}^{(1)}_{\text{z gy}} = \mu B \, \boldsymbol{\nabla} \cdot \widehat{\mathbf{b}} + \frac{\mu c}{e B_\parallel^*} p_z \widehat{\mathbf{b}} \times \left(\widehat{\mathbf{b}} \times \frac{\boldsymbol{\nabla} \times \mathbf{B}}{B} \right) \cdot \boldsymbol{\nabla} B \tag{B10}$$

$$- e \boldsymbol{\nabla} \left\langle \psi_{1\text{gc}} \right\rangle \cdot \left(\widehat{\mathbf{b}} - \frac{c}{e B_\parallel^*} p_z \mathbf{G} \right) \tag{B11}$$

$$\equiv \text{dvapdt0} + \text{dvapdt1} \tag{B12}$$

where we have used the divergence free property of magnetic field: $\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla} B = -B \, \boldsymbol{\nabla} \cdot \widehat{\mathbf{b}}$. We have also organized the terms in two groups: the unperturbed guiding-center contributions dvapdt0 and those containing linear gyro averaged electromagnetic potential dvapdt1:

$$\left\langle \psi_{1\text{gc}} \right\rangle = \left\langle \phi_{1\text{gc}} \right\rangle - \frac{1}{c} \frac{p_z}{m} \left\langle A_{1\parallel gc} \right\rangle . \tag{B13}$$

Appendix C: Noether's method

In this section we sketch the main steps of the Noether's method for systematic conservation laws derivation.

Our main goal is to write the Noether equation:

$$\frac{\partial \mathsf{S}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = \delta \mathcal{L}^{\mathcal{E}} \tag{C1}$$

First of all we need to collect all the exact derivatives terms (Noether's terms)

$$\begin{aligned} 0 \;=\; & -\int d^8 Z \left\{ \mathcal{SH}_1, \mathcal{F} \right\}_{\text{ext}} \\ & - \int \frac{d^4 x}{4\pi} \boldsymbol{\nabla} \cdot \left(\mathbf{E}_1 \; \delta\phi_1 \right) - \int \frac{d^4 x}{4\pi} \frac{1}{c} \, \partial_t \left(\mathbf{E}_1 \; \cdot \delta \mathbf{A}_1 \right) \\ & + \int \frac{d^4 x}{4\pi} \boldsymbol{\nabla} \cdot \left(\left(\mathbf{B}_0 + \epsilon \mathbf{B}_1 \right) \times \delta \mathbf{A}_1 \right) \end{aligned} \tag{C2}$$

The main idea is to identify the partial time derivatives (density terms) and partial space derivatives (flux terms). As we can see, three last terms obtained from the Maxwell part of the action functional can be already identified as flux and density terms.

Terms, obtained from the Vlasov part of the Eulerian action functional require some manipulations. First of all we explicitly write the expression for the Poisson bracket:

$$\begin{aligned} & - \int d^8 Z \left\{ \mathcal{SH}, \mathcal{F} \right\}_{\text{ext}} = \int d^8 Z \left\{ \mathcal{F}, \mathcal{SH} \right\}_{\text{ext}} \equiv \int d^8 Z \left(\frac{\partial}{\partial z^a} \mathcal{F} \right) J^{ab} \frac{\partial}{\partial z^b} \left(\mathcal{SH} \right) \\ & = \int d^8 Z \; J^{ab} \; \frac{\partial}{\partial z^a} \left(\mathcal{F} \frac{\partial}{\partial z^b} \left(\mathcal{SH} \right) \right) - \int d^8 Z \; J^{ab} \;\; \mathcal{F} \; \frac{\partial^2}{\partial z^a \partial z^b} \left(\mathcal{SH} \right) \end{aligned} \tag{C3}$$

where J^{ab} denotes the Poisson matrix, which is antisymmetric, therefore the last term is equal to zero: the second derivative is symmetric.

At the next step we are using the Liouville theorem of the phase space volume conservation: $\boldsymbol{\nabla} \cdot \dot{z}^a = 0$, with $J = \det \left| J^{ab} \right|$ is the determinant of the Poisson matrix; and $\dot{z}^a = J^{ab} \frac{\partial H}{\partial z^b}$ is the Hamiltonian flow:

$$0 = \frac{1}{J} \frac{\partial}{\partial z^a} \left(J \dot{z}^a \right) = \frac{1}{J} \frac{\partial}{\partial z^a} \left(J J^{ab} \frac{\partial H}{\partial z^b} \right) = \underbrace{J^{ab} \frac{\partial^2 H}{\partial z^a \partial z^b}}_{*} + \underbrace{\frac{1}{J} \frac{\partial}{\partial z^a} \left(J \; J^{ab} \right) \frac{\partial H}{\partial z^b}}_{**}, \tag{C4}$$

the term $*$ is equal to zero because it results multiplication of antisymmetric Poisson matrix and symmetric second partial derivative. Therefore, the remaining $**$ term represents the Liouville identity of the phase space volume conservation.

We need that identity for further Noether's terms manipulation, let us rewrite the non-zero term of (C3):

$$\int d^8Z\ J^{ab}\ \frac{\partial}{\partial z^a}\left(\mathcal{F}\frac{\partial}{\partial z^b}\left(\mathcal{SH}\right)\right)\ =\ \int d^8Z\ \frac{1}{J}\ \frac{\partial}{\partial z^a}\left(J\ \mathcal{F}\ \underbrace{J^{ab}\ \frac{\partial}{\partial z^b}\mathcal{SH}}_{\equiv\{z^a,(\mathcal{SH})\}_{\rm ext}}\right) \tag{C5}$$
$$-\ \int d^8Z\ \mathcal{F}\ \underbrace{\frac{1}{J}\ \frac{\partial}{\partial z^a}\left(J\ J^{ab}\right)\frac{\partial}{\partial z^b}\left(\mathcal{SH}\right)}_{=0},$$

here the last term is equal to zero due to the Liouville identity, with substituting $\mathcal{SH}$ into the term **.

Therefore, we have rewritten Vlasov contribution to Noether's method as follows:

$$-\int d^8Z\left\{\mathcal{SH},\mathcal{F}\right\}_{\rm ext}=\int d^8Z\frac{1}{J}\frac{\partial}{\partial z^a}\left(J\mathcal{F}\left\{z^a,\mathcal{SH}\right\}_{\rm ext}\right) \tag{C6}$$

We are now writing the explicit expression for the phase-space volume element $d^8Z\ \equiv\ J\ d^4x\ d^4p \equiv J\ d^3Xdt\ d^3p\ dw$ and with introducing the quadric vectors for the energy-momentum $p^\nu \equiv (w, p^i)$ and space-time $x^\mu \equiv (ct, X^j)$:

$$\int d^8Z\frac{1}{J}\frac{\partial}{\partial z^a}\left(J\mathcal{F}\left\{z^a,\mathcal{SH}\right\}_{\rm ext}\right)=\int d^4x\ \underbrace{\int d^4p\ \frac{\partial}{\partial p^\nu}\ \left(J\mathcal{F}\left\{p^\nu,\mathcal{SH}\right\}_{\rm ext}\right)}_{=0}$$
$$+\ \int d^4x\ \int d^4p\ \frac{\partial}{\partial x^\mu}\ \left(J\mathcal{F}\left\{x^\mu,\mathcal{SH}\right\}_{\rm ext}\right)=\int d^4x\ \frac{\partial}{\partial x^\mu}\ \int d^4p\ \left(J\mathcal{F}\left\{x^\mu,\mathcal{SH}\right\}_{\rm ext}\right),$$

here the term with energy-momentum derivatives cancels as it integrates the exact derivative; the term which contains the space-time derivatives can be rewritten with taking derivative out of integral, it allows to exchange the energy-momentum integral and the space-time derivative.

We are now ready to proceed with separation of the density and flux contributions from the Vlasov terms to the Noether's equation. We take into account that:

$$\left\{x^\mu,\mathcal{SH}\right\}=\left\{x^\mu,\mathcal{S}\right\}\mathcal{H}+\left\{x^\mu,\mathcal{H}\right\}\mathcal{S}, \tag{C7}$$

therefore with using definition of the extended Vlasov field $\mathcal{F}\ \equiv\ \ F\ \delta(w-H)$ and those of the Hamiltonian over the extended phase space $\mathcal{H}\equiv H-w$, we have by definition of the δ-function:

$$\int dw\ \mathcal{F}\left\{x^\mu,\mathcal{S}\right\}\mathcal{H}\equiv\int dw\ \delta(w-H)F\left\{x^\mu,\mathcal{S}\right\}(H-w)=0. \tag{C8}$$

Finally, only contributions of the remaining term count:

$$\frac{\partial}{\partial x^\mu}\int d^4p\ \mathcal{F}\left\{x^\mu,\mathcal{H}\right\}\mathcal{S}=\frac{1}{c}\frac{\partial}{\partial t}\int d^4p\ \mathcal{F}\ \mathcal{S}\underbrace{\left\{c\ t,\mathcal{H}\right\}}_{\equiv c}+\boldsymbol{\nabla}\cdot\int d^4p\ \mathcal{F}\ \mathcal{S}\ \underbrace{\left\{X^i,\mathcal{H}\right\}}_{\equiv\dot{X}_i} \tag{C9}$$

Collecting now expressions for density contributions to the Noether's equation from Maxwell's part and both Vlasov parts of the Eulerian action, we get:

$$\mathsf{S} \equiv -\frac{\epsilon^2}{4\pi c}\left(\mathbf{E}_1 \cdot \delta\mathbf{A}_1\right) + \int d^4p\ \mathcal{F}\ \mathcal{S}. \tag{C10}$$

In the same time for the flux part we have:

$$\mathbf{J} = -\frac{\epsilon^2}{4\pi}\delta\phi_1\ \mathbf{E}_1 - \frac{\epsilon}{4\pi}\left[\delta\mathbf{A}_1 \times (\mathbf{B}_0 + \epsilon\mathbf{B}_1)\right] + \int d^4p\ \mathcal{F}\ \mathcal{S}\{\mathbf{X}, H_1\} \tag{C11}$$

As we have mentioned in section III C, we are concentrating on the energy conservation derivation for the second order Gyrokinetic Maxwell-Vlasov system.

d. First order gyrocenter displacement

From the general reduction procedure at the first order in ϵ_δ we know the expression for the lowest order gyrocenter displacement:

$$\boldsymbol{\rho}_1 = -\left\{S_1, (\mathbf{X} + \boldsymbol{\rho}_0)\right\}_{\mathrm{gc}}, \tag{C12}$$

where S_1 is the lowest order gyrocenter transformation generating function.

From the general reduction procedure, we have:

$$H_{1\mathrm{gy}} \equiv e\ \psi_1(\mathbf{X} + \boldsymbol{\rho}_0) - \left\{S_1, H_{\mathrm{gc}}\right\}_{\mathrm{gc}}, \tag{C13}$$

The expression for S_1 can be also obtained from the condition that the gyrophase dependent part of the linear electromagnetic perturbation $\widetilde{\psi}_1$ is removed from the lowest order gyrocenter Hamiltonian:

$$H_{1\mathrm{gy}} = e\ \psi_1(\mathbf{X} + \boldsymbol{\rho}_0) - e\ \widetilde{\psi}_1(\mathbf{X} + \boldsymbol{\rho}_0) \tag{C14}$$

then with considering only the lowest order contribution to the guiding-center Poisson bracket and taking into account that H_{gc} is gyrophase independent, we have the equation, which defines the gyrocenter generating function at the first order.

$$\frac{\Omega}{B}\ \frac{\partial S_1}{\partial\theta}\ \frac{\partial H_{\mathrm{gc}}}{\partial\mu} = e\ \widetilde{\psi}_{1\mathrm{gc}}, \tag{C15}$$

and therefore at the lowest order:

$$\frac{dS_1}{d\theta} = \frac{e}{\Omega}\ \widetilde{\psi}_{1\mathrm{gc}}, \tag{C16}$$

and the lowest order generating function is given by:

$$S_1 = \frac{e}{\Omega} \int^{\theta} d\theta \; \widetilde{\psi}_{1\mathrm{gc}}. \tag{C17}$$

This demonstrates a tight link between definition of the reduced particle position (new polarization displacement $\boldsymbol{\rho}_1$) and the elimination of the gyrophase dependency from the reduced Hamiltonian dynamics. Now, we can explicitly calculate the expression for the gyrocenter displacement. We substitute the expression for the first order gyrocenter generating function S_1, given by (C17) into the equation (C12).

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