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D Brunetti et al.

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Ideal and resistive magnetohydrodynamic instabilities in a cylindrical plasma with a sheared axial flow

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The problem of the linear stability of internal magnetohydrodynamic (MHD) modes in a cylindrical plasma with a sheared axial flow is addressed. A Newcomb's-like equation describing the perturbation is derived and exactly solved for a class of analytic profiles for rotational transform, equilibrium flow and pressure. A dispersion relation for ideal modes is then derived and analysed for different limits of the poloidal mode number (viz. $m = 1$, $m > 1$ and $m \gg 1$). In the resistive case, a simple and exact expression for the tearing stability index Δ' is derived using the same class of equilibrium profiles. It is found that a small flow shear have a destabilising effect, while if the flow shear is dominant over the magnetic shear the tearing mode is stabilised. Implications on the stability of the $m = 1$ resistive mode are also discussed.

I. INTRODUCTION

The theory of the MHD stability properties of rotating plasmas is scattered over fifty years, spanning from astrophysical [1] to geophysical [2] research. Plasma rotation is an important line of resarch also in the fusion plasmas framework. Significant plasma toroidal rotation (either intrinsic or induced e.g. with neutral beam injection) and flow shear, are commonly present in modern tokamak experiments. Plasma rotation can strongly affect the plasma dynamics, e.g. having an important role in the suppression of turbulence [3, 4]. In particular rotation can stabilise various types of MHD instabilities such as resistive wall modes or tearing modes [5, 6].

Therefore great effort has been expended to assess the behaviour of MHD instabilities (ideal and resistive) when the equilibrium plasma rotation is considered. The presence of equilibrium flows complicates the structure of the MHD equations since the self adjointness of the force operator is lost as shown in the classical paper by Frieman and Rotenberg [7]. Nevertheless a sufficient condition for stability can be derived with the energy principle formalism [7].

The approach presented in Ref. [7] has been extensively applied for the study of highly localised (Suydam-like) modes in cylindrical and toroidal ideal plasmas. Various stability criteria have been derived with both poloidal and toroidal rotation [8–10]. Global $m = 1$ modes, i.e. internal kinks, have also been analytically investigated in cylindrical and toroidal geometry [11–13]. Sonic and subsonic toroidal rotation is found to gyroscopically stabilise the internal kink mode [12, 13], while poloidal rotation stabilises the $m = 1$ internal kink through a *Velikhov* mechanism [1, 11].

More recently the impact of sheared toroidal flows on quasi-intechange and Kelvin-Helmholtz like instabilities has been adressed both analytically and numerically in toroidal plasmas with a central region of low magnetic shear [14–16]. While stabilisation can occur for quasi-interchange modes when a rigid plasma rotation exceeds a particular threshold [14], Kelvin-Helmholtz like global modes can be driven unstable by the rotation shear when the rotation frequency approaches a significant fraction of the Alfén frequency [15, 16].

Resistive modes can also be strongly affected by rotational effects. Rigid rotation effects have been taken into account in cylindrical geometry for the problem of reconnecting low m modes in Ref. [17]. Such a flow does not have an impact on the stability properties since it introduces only a Doppler shift in the rotation frequencies of the perturbations. Resistive interchange modes and the resistive $m = 1$ internal kink have been analysed in Ref. [18] in a poloidally rotating plasma. It has been found that the magnetic well (therefore the stability) is modified by the Velikhov effect depending of the shape of the rotation profile around the resonant point. In this work a global solution for the particular case of constant magnetic well has been obtained. The effects of sheared flows on the linear dynamics of tearing modes have been analysed analytically and numerically mostly in slab geometry [19–22]. The modification of the tearing stability parameter Δ' [23] has been neglected in Ref. [21], while in Ref. [22] it is shown that even a weak plasma flow can have a large impact on Δ'. More recently a numerical analysis of linear and nonlinear dynamics of tearing modes in cylindrical geometry with axial and poloidal flows has been performed [24, 25]. In these studies it is shown that Δ' is strongly affected by the global structure of the flow. In particular it is found that purely sheared axial/poloidal flows have a destabilising/stabilising effect on tearing modes independently of the sign of the flow (while for helical flows this symmetry breaks). Finally, in Ref. [26] an expression for Δ' has been derived in toroidally geometry using the WKB approximation ($m \gg 1$, see Ref. [23]).

This work is thus motivated by giving a unified ap-

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proach for the stability analysis of MHD perturbations (both ideal and resistive) in an axially flowing plasma. The analysis has been performed focusing on a particular class of analytic profiles (for rotational transform, axial flow and pressure) for which the treatment of the magnetic perturbation proves to be exact (note that in contrast with Ref. [18] we do not assume a constant magnetic well). Nonetheless it is envisaged that the same conclusions hold even for more generic equilibrium profiles. This approach turns out to be advantageous both for ideal and resistive modes. In the first case, when ideal pertubations are treated, we are not restricted to focus to a particular m at the time, but rather it is possible to derive a single general dispersion relation which is analysed considering the various limiting cases of the poloidal mode number m. In the second case, having the exact form of the magnetic perturbation with the inclusion of the toroidal flow, allows us to derive a neat and rather simple expression for the tearing stability index Δ' (previous works either dealt with a slab geometry[22] or found approximate solutions [26]).

The paper is organised as follows: in Section II we describe the geometry and physical model of the system under consideration as well as the characteristics of the equilibrium configuration. In Section III the perturbed physical quantities are calculated and a Newcomb's like equation is derived with the inclusion of sheaed axial flow and finite β effects. Section IV is devoted to the analysis of ideal perturbations. The solution of the eigenmode equation is obtained and the dispersion relation is eventually derived and analysed for different limits of the poloidal mode number m. Section V deals with the effects of the flow shear on the tearing stability parameter Δ' while flow shear effects in the resistive layer are excluded. A brief discussion on the flow shear effects on the $m = 1$ mode is presented. Finally the findings of this work and future outlook are summarised in section VI.

II. PHYSICAL MODEL

We define a cylindrical coordinate system (r, ϑ, ζ) where the variable r labels flux (isobaric) surfaces, ϑ is the poloidal angle and ζ is the longitudinal coordinate. It is useful to introduce the variable $\varphi = \zeta/R$ where $R = L/2\pi$ and L is the cylinder length [27]. Note that φ can be interpreted as a toroidal-like variable. Thus in our analysis we will use the coordinate system defined by the variables (r, ϑ, φ). The radius of the plasma cylinder is denoted by a. We work in *large aspect ratio* approximation, i.e. we assume that $a/R \ll 1$. It is assumed that the plasma is surrounded by a perfectly conducting metallic wall, so that perturbations are forced to vanish at the plasma edge (i.e. at $r = a$).

In cylindrical geometry the covariant metric tensor coefficients are then given by $g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, $g_{\varphi\varphi} = R^2$, $g_{r\vartheta} = g_{r\varphi} = g_{\vartheta\varphi} = 0$ and $\sqrt{g} = rR$. The contravariant and covariant components of a vector $\boldsymbol{A}$ are given by $A^i = \boldsymbol{A} \cdot \boldsymbol{\nabla} q^i$ and $A_i = \boldsymbol{A} \cdot \boldsymbol{e}_{q^i}$, $\boldsymbol{\nabla} q^i$ and $\boldsymbol{e}_{q^i}$ being the contravariant and covariant basis vectors respectively ($i = r, \vartheta, \varphi$ and $q^i = (r, \vartheta, \varphi)$). For a vector $\boldsymbol{A}$, it is useful to introduce the physical cylindrical projections, i.e.:

$$A_p = \frac{\boldsymbol{A} \cdot \boldsymbol{\nabla}\vartheta}{|\boldsymbol{\nabla}\vartheta|} = rA^\vartheta, \tag{1}$$

$$A_z = \frac{\boldsymbol{A} \cdot \boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} = RA^\varphi. \tag{2}$$

Note that since in cylindrical geometry $g_{rr} = g^{rr} = 1$, the radial covariant and contravariant components of the vector field correspond to the physical radial component of the field itself. Therefore we shall denote the physical radial componend of the field with its covariant radial projection A_r.

The magnetic field is written in terms of the usual flux functions [28, 29]:

$$\boldsymbol{B} = \boldsymbol{\nabla} F \times \boldsymbol{\nabla}\vartheta - \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\varphi. \tag{3}$$

We approximate $B^\varphi = B_0^\varphi \approx const$ valid for low-β cylindrical plasmas [27]. Hence the equilibrium toroidal flux is given by the expression $F_0' = rB_0$, where B_0 is the physical toroidal field assumed constant and a prime indicates a derivative wrt the radial variable. The safety factor is $q = F'/\psi'$ and its inverse is $\mu = 1/q$. The plasma current density is $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$ (having normalised $\mu_0 = 1$).

Our stability analysis, presented in the next sections, is based on the following ideal incompressible MHD equations [30]:

$$\varrho\left[\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v}\right] = -\boldsymbol{\nabla} p + \boldsymbol{J} \times \boldsymbol{B}, \tag{4}$$

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{5}$$

$$\partial_t p + \boldsymbol{v} \cdot \boldsymbol{\nabla} p = 0, \tag{6}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{7}$$

where $\boldsymbol{v}$ is the plasma MHD velocity, p the plasma pressure and ϱ the mass density which, for sake of simplicity, is assumed constant both in space and time.

The plasma equilibrium is thus given by the force balance equation where the time dependencies are set to zero:

$$\varrho \boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\boldsymbol{v}_0 = -\boldsymbol{\nabla} p_0 + \boldsymbol{J}_0 \times \boldsymbol{B}_0. \tag{8}$$

We formally retain the convective term since we impose an equilibrium axial flow $\boldsymbol{v}_0 = \Omega(r)\boldsymbol{e}_\varphi$ (for sake of simplicity we set $\Omega \equiv \Omega(r)$) which is assumed to be of the same order of magnitude of the plasma pressure. This implies a Mach number of order unity ($\mathscr{M} \sim \sqrt{\varrho(R_0\Omega)^2/p}$) [12]. It is useful to stress the point that because of the cylindricity of the problem, the isobaric surfaces always correspond to the flux surfaces regardless of the strength of the equilibium flow, i.e. also

when $\mathscr{M} > 1$ (this is because $\boldsymbol{v}_0 \cdot \boldsymbol{\nabla} \boldsymbol{v}_0 = 0$). Hence the radial projection of (8), i.e. the Grad-Shafranov equation, is given by:

$$q\beta' = -\tfrac{r}{R} J_0^{\varphi} / B_0^{\varphi}, \tag{9}$$

where $\beta = p_0 / B_0^2$, having used $J_0^{\vartheta} = 0$ (this is because $B_0' = 0$).

III. NEWCOMB EQUATION WITH FLOW SHEAR AND β EFFECTS

A. Description of the perturbed fields

We adopt an Eulerian point of view for the description of the perturbation, which appears to be simpler to handle compared to the standard Lagrangian approach with the Frieman-Rotenberg equations [7]. The Frieman-Rotenberg equations are particularly useful when a δW approach (viz. energy principle) is adopted, which is not used in our analysis. Hereafter we implicitely assume that the tilde denotes the perturbations.

The time and angular dependence of the perturbation is taken of the form $\exp(-i\omega t + im\vartheta - in\varphi)$. We assume that different Fourier modes are decoupled, hence the analysis considers a single Fourier harmonic with mode number (m, n). From (6), after linearisation, we have the expression for the perturbed pressure which reads:

$$\tilde{p} = -ip_0' \tilde{v}^r / \hat{\omega}, \tag{10}$$

where we defined the Doppler shifted frequency:

$$\hat{\omega} = \omega + n\Omega. \tag{11}$$

The linearised contravariant radial, poloidal and toroidal projections of (5) lead respectively to ($' \equiv d/dr$):

$$\sqrt{g}\tilde{B}^r = -rB_0 k_{||} \xi^r, \tag{12}$$

$$\sqrt{g}\tilde{B}^{\vartheta} = -irB_0 k_{||} \xi^{\vartheta} - irB_0 \mu' \xi^r, \tag{13}$$

$$\sqrt{g}\tilde{B}^{\varphi} = -irB_0 k_{||} \xi^{\varphi} + i\frac{\Omega'}{\hat{\omega}} (\sqrt{g}\tilde{B}^r), \tag{14}$$

where $\xi^i = \tilde{v}^i / \hat{\omega}$ $(i = r, \vartheta, \varphi)$ and $k_{||} = m\mu - n$. Linearising the covariant poloidal projection of (4) yields:

$$\varrho \hat{\omega} \tilde{v}^{\vartheta} = -inqp_0' \tilde{v}^r / \hat{\omega} + B_0^{\varphi} (m\tilde{B}_{\varphi} + n\tilde{B}_{\vartheta}), \tag{15}$$

which at leading order can be rearranged in the following form:

$$\sqrt{g}\tilde{B}^{\varphi} = iq\tfrac{n}{m}\beta' rB_0 \xi^r - \tfrac{n}{m}\tfrac{r^2}{R^2}(\sqrt{g}\tilde{B}^{\vartheta}), \tag{16}$$

where $\omega_A = B_0 / (R\sqrt{\varrho})$. Note that the same expression can be obtained at leading order by taking the covariant φ projection of (4). Finally the incompressibility condition (7) is written as:

$$\tfrac{1}{r}(r\hat{\omega}\xi^r)' + i\hat{\omega}(m\xi^{\vartheta} - n\xi^{\varphi}) = 0. \tag{17}$$

The set of equations (12)-(14), (16) and (17) are used to express $\tilde{B}^i$ $(i = r, \vartheta, \varphi)$, $\tilde{v}^{\vartheta}$ and $\tilde{v}^{\varphi}$ as functions of $X = i\xi^r$. We assume that $\beta \sim \mathscr{O}(\varepsilon^2)$ and $\omega/\omega_A \sim \Omega/\omega_A \sim \mathscr{O}(\varepsilon)$. Eventually we obtain for the magnetic field components:

$$\sqrt{g}\tilde{B}^r = irB_0 k_{||} X, \tag{18}$$

$$\sqrt{g}\tilde{B}^{\vartheta} = -\frac{B_0}{m}[rk_{||}X]' + \frac{n}{m}(\sqrt{g}\tilde{B}^{\varphi}), \tag{19}$$

$$\sqrt{g}\tilde{B}^{\varphi} = \frac{nB_0}{m}\{\tfrac{1}{m}\left(\tfrac{r}{R}\right)^2 [rk_{||}X]' + rq\beta' X\}. \tag{20}$$

Note that we obtain the same result also in the case $m \sim n \sim 1/\varepsilon$ (this will be useful when we will study high-m localised, i.e. Suydam, modes).

By means of (9), equation (20) is cast in the form:

$$\sqrt{g}\tilde{B}^{\varphi} = \tfrac{k_z r^2}{m} J_z X - \tfrac{k_z r^2}{m^2}[(mB_p + rk_z B_z)X]', \tag{21}$$

where $k_z = -n/R$, B_p and $B_z = B_0$ are the equilibrium physical poloidal and toroidal magnetic fields respectively while J_z is the physical equilibrium toroidal current density (cf. (1) and (2)). It can be easily shown that [27]:

$$\tilde{B}_z = R\tilde{B}^{\varphi} = -\frac{1}{r}[\partial_{\vartheta} Y + (rB_z X)'], \tag{22}$$

where Y is given by:

$$Y = \frac{i}{m}(rB_z X)' + \frac{ik_z r^2}{m^2} J_z X - \frac{ik_z r^2}{m^3}[(mB_p + rk_z B_z)X]'. \tag{23}$$

Analogously, a very simple manipulation shows that:

$$\tilde{B}_p = r\tilde{B}^{\vartheta} = -(B_p X)' + \tfrac{1}{R}\partial_{\varphi} Y, \tag{24}$$

and finally:

$$\tilde{B}_r = \tilde{B}^r = \tfrac{1}{r}(B_p \partial_{\vartheta} + \tfrac{rB_z}{R}\partial_{\varphi})X. \tag{25}$$

It is possible to show that for the next calculation only the leading order of ξ^{ϑ} is required. Thus we approximate:

$$\xi^{\vartheta} = \frac{i}{mr}(r\xi^r)'. \tag{26}$$

Equations (22)-(26) will be used in the next section where the Newcomb equation for axially flowing plasmas is derived.

B. Derivation of the Newcomb equation

Having the expressions for all the pertuberd quantities we can proceed in the derivation of the Newcomb equation with flow shear. This is obtained by applying the operator $\frac{1}{B_0^\varphi}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times$ to the momentum balance equation (4).

The left hand side of the resulting expression reads:

$$\frac{1}{B_0^\varphi}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times\{\varrho\,[\partial_t\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}]\} = \frac{\varrho}{B_0^\varphi}\left[(\partial_t+\Omega\partial_\varphi)\tilde{w}^\varphi+\tfrac{R}{r}\Omega'\partial_\vartheta\tilde{v}^\varphi-\Omega'\tilde{w}^r\right]$$

where $\boldsymbol{w}=\boldsymbol{\nabla}\times\boldsymbol{v}$ and $\boldsymbol{w}_0=\boldsymbol{\nabla}(R^2\Omega)\times\boldsymbol{\nabla}\varphi$. Making use of the relation $\tilde{w}^r=\frac{1}{rR}[R^2\partial_\vartheta\tilde{v}^\varphi-r^2\partial_\varphi\tilde{v}^\vartheta]$ we can eliminate the $\tilde{v}^\varphi$ dependence in the relation above so that the final result reads:

$$\frac{1}{B_0^\varphi}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times\{\varrho\,[\partial_t\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}]\} = \frac{\varrho}{B_0^\varphi}\left[(\partial_t+\Omega\partial_\varphi)\tilde{w}^\varphi+\frac{r}{R}\Omega'\partial_\varphi\tilde{v}^\vartheta\right]. \quad (27)$$

Since $\tilde{w}^\varphi=\frac{1}{rR}[\partial_r r^2\tilde{v}^\vartheta-\partial_\vartheta\tilde{v}^r]$, only $\tilde{v}^r$ and $\tilde{v}^\vartheta$ are required in Eq. (27). Operating the substitution $\partial_\varphi\to -in$ and $\partial_t\to -i\omega$, a rather lenghty but straightforward manipulation gives:

$$R\hat{\omega}[\tilde{w}^\varphi+\frac{r}{R}n\Omega'\xi^\vartheta] = \frac{1}{mr^2}\times \times\left[\left(r^3\hat{\omega}^2X'\right)'+r(1-m^2)\hat{\omega}^2X+2r^2\hat{\omega}\hat{\omega}'X\right], \quad (28)$$

where in the derivation of the expression above we made use of the relation $\hat{\omega}'=n\Omega'$.

We now note that:

$$\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times[\tilde{\boldsymbol{J}}\times\boldsymbol{B}_0+\boldsymbol{J}_0\times\tilde{\boldsymbol{B}}] = \boldsymbol{B}_0\cdot\boldsymbol{\nabla}\tilde{J}^\varphi+\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times(\boldsymbol{J}_0\times\tilde{\boldsymbol{B}}). \quad (29)$$

Using (18) and (19) it is easy to show that:

$$\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\frac{\tilde{J}^\varphi}{B_0^\varphi} = \frac{1}{B_zr^2}(B_p\partial_\vartheta+\tfrac{r}{R}B_z\partial_\varphi)\times \times\left\{\partial_r[r(\tfrac{1}{R}\partial_\varphi Y-[B_pX]')]-\tfrac{1}{r}\partial_\vartheta[(B_p\partial_\vartheta+\tfrac{r}{R}B_z\partial_\varphi)X]\right\}. \quad (30)$$

We have also the following relation:

$$1/B_0^\varphi\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times(\boldsymbol{J}_0\times\tilde{\boldsymbol{B}}) = \frac{1}{B_zr^2}\left\{\frac{rB_z}{R}\partial_r(rJ_z\partial_\varphi X)+\partial_\vartheta(\frac{rJ_z}{R}\partial_\varphi Y)+[B_p(rJ_z)'+p_0']\partial_\vartheta X\right\}. \quad (31)$$

Collating the results given by Eqs. (30) and (31), the action of the operator $\frac{1}{B_0^\varphi}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times$ on the right hand side of (4) reads (cf. Ref. [27]):

$$\frac{1}{B_0^\varphi}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\times[\tilde{\boldsymbol{J}}\times\boldsymbol{B}_0+\boldsymbol{J}_0\times\tilde{\boldsymbol{B}}] = -\frac{i}{mr^2B_z}\left\{\frac{d}{dr}[r(mB_p-\tfrac{rn}{R}B_z)\frac{dX}{dr}]-rUX\right\}, \quad (32)$$

where the function U is given by [27, 31, 32]:

$$U = \frac{1}{r^2}(m^2-1+k_z^2r^2)(mB_p+rk_zB_z)^2 + 2k_z^2rp_0'+\frac{2k_z^2}{m^2}(r^2k_z^2B_z^2-m^2B_p^2).$$

Therefore combining Eqs. (28) and (32) the Newcomb equation [31] with flow shear is eventually obtained (we recall that $\hat{\omega}$ is given by (11)):

$$\frac{d}{dr}\left[r^3(\omega_A^2k_{||}^2-\hat{\omega}^2)\frac{dX}{dr}\right]-\omega_A^2\frac{rR^2}{B_z^2}UX + r\hat{\omega}^2(m^2-1)X-2r^2\hat{\omega}\hat{\omega}'X=0, \quad (33)$$

where in cylindrical geometry $\mu=RB_p/rB_z$. The function U is then well approximated by the following expression [27]:

$$\frac{R^2}{B_z^2}U\approx(m^2-1)(m\mu-n)^2+2n^2\frac{rp_0'}{B_z^2}+4\frac{n^3}{m^2}\frac{r^2}{R^2}(n-m\mu). \quad (34)$$

Equation (33) together with (34) will be solved in the next sections for a particular class of analytic profiles for μ, Ω and p_0, where the problem of ideal and resistive internal modes is addressed.

IV. IDEAL INTERNAL MODES

This section is devoted to the solution of Eq. (33). Before embarking on the derivation of the solution of the eigenmode equation, we shall note that because of the presence of the equilibrium flow there are three singular points which have to be treated carefully. The first dangerous point is where $k_{||}=0$ (cf. Eq. (18)), i.e. the usual singularity in the fluid displacement X. From equation (33) it is clear that two additional singularities occur where $k_{||}^2-(n\Omega/\omega_A)^2=0$ [22].

In the particular case of a rigid flow along the axis of the cylinder, the stability properties of the mode should not be affected. Indeed if the analysis is carried in a reference frame which is moving along the axis with the same velocity [17, 22], the problem of having additional singularities is avoided. This is because the flow effects enter simply through a Doppler shift of the eigenfrequencies [17] (note that in the comoving reference frame the eigenfrequencies do not have an imaginary part).

The case of a nonuniform flow is generally more difficult to handle mathematically. However a great simplification can be obtained by chosing a reference frame

in which the equilibrium axial flow profile vanishes at the point where $k_{||} = 0$ denoted by r_s [21, 22].

If we multiply equation (33) by X^* (i.e. the complex conjugate of X) and then integrate from 0 to a, a quadratic equation for ω is obtained (similar to Eq. (29) in Ref. [7]), whose solution is:

$$\omega = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{35}$$

For sake simplicity we use the normalisation:

$$\omega_A = 1.$$

The coefficients A, B and C are given by:

$$A = \int_0^a r\{r^2|X'|^2 + r(m^2-1)|X|^2\}dr,$$
$$B = 2n\int_0^a r\{\Omega r^2|X'|^2 + [\Omega(m^2-1) - r\Omega']|X|^2\}dr,$$
$$C = -\int_0^a r\{r^2(k_{||}^2 - n^2\Omega^2)|X'|^2 + [\tfrac{R^2}{B_z^2}U - n^2\Omega^2 + 2n^2 r\Omega\Omega']|X|^2\}dr.$$

Note that in the case of no flow ($\Omega = 0$) the term B vanishes and the usual results from ideal normal modes analysis are obtained [33].

Let us first consider modes with $m \gg 1$, i.e. Suydam modes. Since we assume that $q \sim 1$ it necessarily follows that also $n \gg 1$. Assume also that these modes are localised around their own resonant surface r_s and have at leading order definite parity, i.e. even or odd. Near r_s we approximate $\Omega \sim r_s\Omega_s' x$ where $x = (r - r_s)/r_s$ and $\Omega' = d\Omega/dr$ (the subscript s indicates that the quantity is evaluated at r_s). Note that in the next sections we will provide an exact expression for the gradient of the flow. We neglect terms proportional to $r_s^2\Omega_s'' x^2$ under the assumption that $r_s^2\Omega_s'' \sim r_s\Omega_s'$. Because of the localisation of these modes, integration is performed over a region of width δr. We approximate the radial derivative as $r_s\frac{d}{dr} \approx \frac{d}{dx}$. Let us order the physical quantities as $\frac{d}{dx} \sim \frac{1}{x} \sim m \sim \frac{1}{\varepsilon}$ and $r_s\Omega_s' \sim \frac{\delta r}{a} \sim \varepsilon$. Using (34) with ordering presented above, it follows that either $\frac{R^2}{B_z^2}U \sim 1$ if $\mu_s' \sim \varepsilon$ or $\frac{R^2}{B_z^2}U \sim \frac{1}{\varepsilon^2}$ is $\mu_s' \sim 1$, where μ_s' measures the magnetic shear. By exploiting the antisymmetry in x of the equilibrium flow profile, the first and second terms in B and the last term in C vanish when integrated. Hence it is immediate to show that $A \sim \frac{1}{\varepsilon}a^2|X|^2$, $B \sim \varepsilon a^2|X|^2$ and $C \sim \varepsilon a^2|X|^2$ at least. Hence we can approximate (35) with $\omega \approx \pm\sqrt{-C/A}$ which shows that ω is either purely real or purely imaginary.

Analogously, we assume that $m \geq 1$ modes with $m \sim 1$ have strong radial excurtions around the resonant point r_s. Hence the contributions to the integrals in the coefficients A, B and C come from two distinct regions. In the region far from r_s where we adopt the following ordering $a\frac{d}{dr} \sim 1$ with the integration introducing only terms of order unity. Conversely close to r_s in a layer of width $\delta r/a\varepsilon$, where $r_s\frac{d}{dr} \approx \frac{d}{dx}$, we assume that $\frac{d}{dx} \sim \frac{1}{x} \sim \frac{1}{\varepsilon}$. In both regions it is assumed that $r_s\Omega_s' \sim \varepsilon$. Thus the main contribution to A comes from the region around r_s and $A \sim \frac{1}{\varepsilon}a^2|X|^2$. It is easy to see that $B \sim \varepsilon a^2|X|^2$ and that the dominant part for the coefficient C comes from the integration far from r_s so that $C \sim (\mu_s')^2 a^2|X|^2$. Therefore for not too small S, i.e. $\mu_s' > \varepsilon$ we have $\sqrt{AC}/B \sim \mu_s'/\varepsilon^{3/2} \gg 1$ indicating that also in this case the eigenfrequencies are at leading order either purely real or purely imaginary.

We therefore identify two regions, one for which $k_{||} \gtrsim 1$ (**external**) and another one for which $k_{||} \ll 1$ (**inertial**). In the external region for sufficiently small growth rates, viz. close to the stability boundary, inertial effects can be neglected while they become important in the inertial region. Each region is treated separately. Matching the solutions in the two regions, eventually leads to the dispersion relation which is then discussed analysing the various limiting cases in m ($m \gg 1$ and $m \geq 1$).

A. External region solution

We assume that $\omega < n\Omega$, so that inertia is neglected. With a careful choice of the profiles, an exact analytic solution of (33) can be easily found. We model the parallel wave vector as [27]:

$$k_{||} = \frac{nS}{\lambda}[1 - (r/r_s)^\lambda], \tag{36}$$

where S is the magnetic shear at r_s, i.e. $S = r_s q_s'/q_s$. Thus the following profile for the equilibrium flow is chosen (the shape of Ω is sketched in Fig. 1):

$$\Omega = \Omega_0[1 - (r/r_s)^\lambda], \tag{37}$$

so that $(rd\Omega/dr)|_{r_s} = -\lambda\Omega_0$. We assume here that:

$$S > \lambda\Omega_0, \tag{38}$$

which means that at r_s the magnetic shear S is larger than the flow shear. This inequality is always satisfied when weak flows are considered. Note that in some particular cases we can relax this constraint (this will be discussed in the next sections). With such a choice for safety factor and equilibrium flow shear, we see that no additional singularities other than the usual $k_{||} = 0$ are introduceded, i.e. the equations are singular only at the point $r = r_s$. It is also assumed that the pressure profile takes the form:

$$p_0 = p_{ax}\left[1 - (r/a)^\lambda\right]. \tag{39}$$

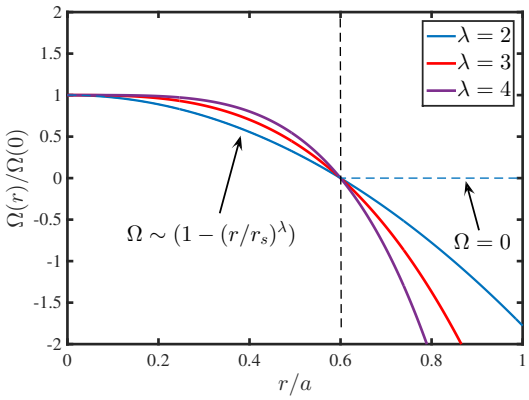

Figure 1. Example of flow profiles given by Eq. (37) with $r_s/a = 0.6$.

By introducing the variable $z = (r/r_s)^\lambda$ we recast Eq. (33) in the following form:

$$\frac{d}{dz}\left[z^{2/\lambda+1}(1-z)^2\frac{dX}{dz}\right] - z^{2/\lambda-1}\{\frac{(m^2-1)}{\lambda^2}(1-z)^2 - P_0 z + \delta z^{2/\lambda}(1-z)\}X + Hz^{2/\lambda}(1-z)X = 0, \quad (40)$$

where $P_0 = \frac{2\lambda\beta_T(r_s/a)^\lambda}{[S^2-\lambda^2\Omega_0^2]}$ with $\beta_T = p_{ax}/B_z^2$, $\delta = -\frac{4(n^2/m^2)(S/\lambda)}{[S^2-\lambda^2\Omega_0^2]}\left(\frac{r_s^2}{R^2}\right)$ and:

$$H = \frac{2\Omega_0^2}{[S^2-\lambda^2\Omega_0^2]}. \quad (41)$$

Equation (40) is exactly solvable for parabolic profiles, viz. $\lambda = 2$. For a generic λ we note that:

$$\frac{\delta}{P_0} \sim \frac{\varepsilon^2}{\beta_T}\frac{S}{\lambda^2 q_s^2}, \qquad \frac{\delta}{H} \sim \frac{\varepsilon^2}{\Omega_0^2}\frac{S}{\lambda^3 q_s^2},$$

having assumed that $(r_s/a)^\lambda \sim 1$, i.e. $\lambda \sim 1$. Let us assume that $S \sim 1$, $\lambda > 1$ and $q_s > 1$. With the standard ordering $\beta \sim \Omega_0^2 \sim \varepsilon^2$ and under the assumption that the quantity $S/(\lambda^2 q_s^2)$ is sufficiently small (this holds in view of the ordering presented above), we can neglect the term proportial to δ. This is also justified since we expect that flow shear effects become important when the magnetic shear weakens.

Thus for $z < 1$ the solution of (40) which is regular on the magnetic axis reads [27, 34, 35]:

$$X_-(z) \sim \frac{z^{(m-1)/\lambda}}{(1-z)^{1+s}}F(\xi-s, \zeta-s, 1+\xi+\zeta; z), \quad (42)$$

where F is the Gauss hypergeometric function (cf. Ref. [36] p. 555), $\xi = (m-\bar{m})/\lambda$, $\zeta = (m+\bar{m})/\lambda$ and:

$$\bar{m} = \sqrt{m^2+8-4\delta+4H}, \quad \text{for } \lambda = 2, \delta \neq 0, \quad (43)$$

$$\bar{m} = \sqrt{m^2+2\lambda+\lambda^2(1+H)}, \quad \text{for } \lambda \text{ generic}, \delta = 0. \quad (44)$$

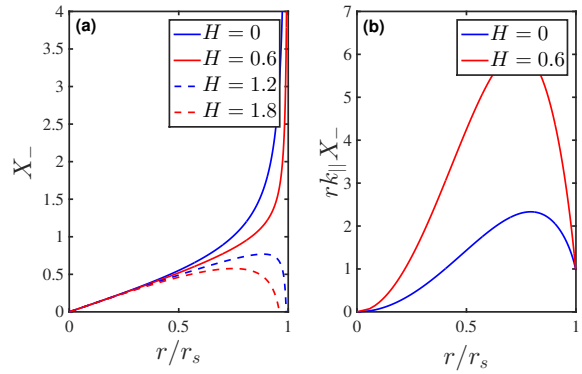

Figure 2. Plot of X_- (a) given by equation (42) and $rk_{||}X_-$ (b) for the $m = 2$ mode for different values of the parameter H defined by (41) with $r_s/a = 1/2$, $\lambda = 2$ and $\delta = 0$. Note that $rk_{||}X_-$ is proportional to the perturbed poloidal flux (see Eq. (A7)) and thus it has been normalised to unity at $r = r_s$. We decided not to plot the quantity $rk_{||}X_-$ for $H > 1$ since X_- has a chance of sign.

Here we have introduced the Suydam parameter:

$$s = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - P_0}, \quad (45)$$

which satisfies $P_0 = -s(s+1)$. We stress the point that the flow shear appears to affect the eigenfunction in the same manner of the effects linear with respect to the magnetic shear (represented by the parameter δ) [27]. In view of the estimates presented above, effects linear wrt the magnetic shear are weaker compared with the flow shear effects. The eigenfunction given by equation (42) is shown in figure 2 where the effects of the flow shear are better visualised.

Conversely when $z > 1$ the solution is:

$$X_+(z) \sim \frac{z^{s-(1+\bar{m})/\lambda}}{(z-1)^{1+s}}\left[F(\zeta-s, -\xi-s, 1-\xi+\zeta; \tfrac{1}{z}) + Dz^{2\bar{m}/\lambda}F(-\zeta-s, \xi-s; 1+\xi-\zeta; \tfrac{1}{z})\right], \quad (46)$$

where the constant D is found by imposing that $X(a) = 0$ giving:

$$D = -\left(\frac{r_s}{a}\right)^{2\bar{m}}\frac{F(\zeta-s, -\xi-s, 1-\xi+\zeta; (r_s/a)^\lambda)}{F(-\zeta-s, \xi-s; 1+\xi-\zeta; (r_s/a)^\lambda)}$$

Using the same notation as in Ref. [27] we have the following asymptotic behaviour of (42) and (46) (we recall that $x = (r-r_s)/r_s$):

$$X \sim \begin{cases} |x|^{-1-s}\left(1+\Delta_c|x|^{1+2s}\right), & \text{if } r < r_s. \\ |x|^{-1-s}\left(1+\Delta_p|x|^{1+2s}\right), & \text{if } r > r_s. \end{cases} \quad (47)$$

where the constants Δ_c and Δ_p are given by:

$$\Delta_c = \lambda^{1+2s}\frac{\Gamma(-1-2s)\Gamma(1+\zeta+s)\Gamma(1+\xi+s)}{\Gamma(1+2s)\Gamma(\xi-s)\Gamma(\zeta-s)}, \quad (48)$$

$$\Delta_p = \lambda^{1+2s}\frac{\Gamma(-1-2s)\Gamma(1+\zeta+s)\Gamma(1-\xi+s)}{\Gamma(1+2s)\Gamma(-\xi-s)\Gamma(\zeta-s)}\times \times\frac{\left[1+D\frac{\Gamma(1+\xi-\zeta)\Gamma(\zeta-s)\Gamma(-\xi-s)}{\Gamma(1-\xi+\zeta)\Gamma(-\zeta-s)\Gamma(\xi-s)}\right]}{\left[1+D\frac{\Gamma(1+\xi-\zeta)\Gamma(1+\zeta+s)\Gamma(1-\xi+s)}{\Gamma(1-\xi+\zeta)\Gamma(1-\zeta+s)\Gamma(1+\xi+s)}\right]}. \quad (49)$$

In the case of vanishing pressure ($s \to 0$), the asymptotic behaviour of (42) and (46) is:

$$X \sim \begin{cases} \frac{1}{|x|} + \hat{\Delta}_c + \lambda\xi\zeta\ln|x|, & \text{if } r < r_s. \\ \frac{1}{|x|} + \hat{\Delta}_p - \lambda\xi\zeta\ln|x|, & \text{if } r > r_s. \end{cases} \quad (50)$$

where we defined the following quantities:

$$\hat{\Delta}_c = \lambda\xi\zeta[\ln\lambda - \Psi(1) - \Psi(2) + \Psi(\zeta+1) + \Psi(\xi+1)] - m + 1, \quad (51)$$

$$\hat{\Delta}_p = \lambda\xi\zeta\ln\tfrac{1}{\lambda} + c_+ + \frac{\hat{D}}{1+\hat{D}}(c_- - c_+), \quad (52)$$

$$c_\pm = \lambda\xi\zeta[\Psi(1) + \Psi(2) - \Psi(\pm\zeta+1) - \Psi(\mp\xi+1)] \mp (\bar{m} \pm 1),$$

with $\hat{D} = D\frac{\Gamma(-\xi)\Gamma(\zeta)\Gamma(1+\xi-\zeta)}{\Gamma(-\zeta)\Gamma(\xi)\Gamma(1-\xi+\zeta)}$ and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function of argument x (cf. Ref. [36] p. 258).

The asymptotics near r_s of the external solution need to be matched with the inertial layer solution which is calculated in the next section.

B. Inertial region

In the inertial region we assume that we are very close to r_s, so that all the term proportional to x in the function U can be dropped, as well as the Ω_s term in (40). Allowing for the inertial correcctions, the equation describing the radial fluid displacement is approximated in the following manner (we recall that $\omega_A = 1$):

$$\frac{d}{dx}\left\{\left[x^2 - 2i\frac{(\gamma/n)r_s\Omega_s'}{S_e^2}x + \frac{\gamma^2/n^2}{S_e^2}\right]\frac{dX}{dx}\right\} + P_0 X = 0, \quad (53)$$

where $\gamma = -i\omega$ and we defined $S_e^2 = S^2 - \lambda^2\Omega_0^2$. The equation above can be reduced to a Legendre differential equation (cf. Ref. [36] p. 331). The two independent solutions X_e, X_o can be expressed in terms of the hypergeometric function as [8]:

$$X_e = F(-\tfrac{s}{2}, \tfrac{1+s}{2}; \tfrac{1}{2}; -\chi^2), \quad (54)$$

$$X_o = \chi F(\tfrac{1-s}{2}, 1+\tfrac{s}{2}; \tfrac{3}{2}; -\chi^2), \quad (55)$$

where s is given by (45) and we introduced the layer variable:

$$\chi = \tfrac{S_e^2}{S}(nx/\gamma) + i\tfrac{\lambda\Omega_0}{S}.$$

Under the assumption $\frac{\gamma}{n} < \frac{S_e^2}{\lambda\Omega_0}x$ (which holds close to the stability boundary) we can approximate $\chi \approx \frac{S_e^2}{S}(nx/\gamma)$. Hence the asymptotic behaviour of (54) and (55) used in the matching region for connecting with the external solution given by (47) are readily found:

$$X_e \sim |\chi|^{-1-s}\left(1 + \Delta_e|\chi|^{1+2s}\right), \quad (56)$$

$$X_o \sim \text{sign}(\chi)|\chi|^{-1-s}\left(1 + \Delta_o|\chi|^{1+2s}\right). \quad (57)$$

where $\Delta_e = \frac{\Gamma(s+1/2)}{\Gamma(-s-1/2)}\frac{\Gamma^2(-s/2)}{\Gamma^2(\frac{1+s}{2})}$ and $\Delta_o = \frac{\Gamma(s+1/2)}{\Gamma(-s-1/2)}\frac{\Gamma^2(\frac{1-s}{2})}{\Gamma^2(1+\frac{s}{2})}$.

If pressure corrections are neglected ($s = 0$), we note that (56) and (57) cannot be matched with the logarithmic term appearing in (50). Thus the inner layer equation must be augmented by an additional term which takes into account the cylindricity of the system [27]. If we write $z = 1 + \lambda x$, including inertial corrections in (40), instead of (53) we have:

$$\frac{d}{dx}\left[(1+\lambda x)^{2/\lambda+1}\left(x^2 + 2i\frac{\lambda\Omega_0}{S_e^2}\frac{\gamma}{n}x + \frac{\gamma^2/n^2}{S_e^2}\right)\frac{dX}{dx}\right] = 0.$$

By writing $X = X_0 + \delta X_1 + \mathscr{O}(\delta^2)$ and $x \sim \hat{\gamma} \sim \delta$, $\frac{d}{dx} \sim \frac{1}{\delta}$ with $\delta \ll 1$, the equation above is expanded in powers of δ and the solution is:

$$X = c_0 + c_1\arctan(\chi) \pm c_3\ln[(\tfrac{\gamma}{n})\left(\tfrac{S}{S_e^2}\right)^2(1+\chi^2)], \quad (58)$$

where $\pm$ stands for $x \lesseqgtr 0$. The asymptotic behaviour, used in the matching region, of the non logarithmic part of (58) is given by (56) and (57) for $s \to 0$ where $\Delta_e \to \infty$ and $\Delta_o \to -\frac{\pi}{2}$. For sufficiently small γ, the logarithmic terms in (58) and (50) are automatically matched with an appropriate choice of the constant c_3.

In the next section, the asymptotics of the inertial layer solutions will be matched with the eigenfunctions obtained in the external region, giving finally the dispersion relation.

C. Dispersion relation for ideal modes

The procedure of matching (47) with (56) and (57), follows the one adopted in Ref. [27] and the final result reads:

$$\hat{\gamma}^{2+4s}\Delta_c\Delta_p - (\Delta_e + \Delta_o)(\Delta_c + \Delta_p)\frac{\hat{\gamma}^{1+2s}}{2} + \Delta_e\Delta_o = 0, \quad (59)$$

where $\hat{\gamma} = \frac{\gamma}{n}\frac{S_e^2}{S}$ with Δ_c and Δ_p given by (48) and (49) respectively. This is the general dispersion relation for a Fourier mode with arbitrary m in a cylindrical plasma with an axial flow. Now we investigate this expression analysing three interesting limits.

1. $m \gg 1$ modes (Suydam-like)

Let us first examine the $m \gg 1$ case with $s \neq 0$. In this limit $\Delta_c = \Delta_p = \Delta_S$ ($D \approx 0$), therefore from (59) we have:

$$\hat{\gamma}^{1+2s} = \Delta_e/\Delta_S, \quad \hat{\gamma}^{1+2s} = \Delta_o/\Delta_S, \tag{60}$$

corresponding to the usual even and odd modes. From (45), writing $s = -\frac{1}{2} + i\alpha$ (where $\alpha = \sqrt{P_0 - 1/4}$) we find that the growth rate is $\hat{\gamma} \sim \exp[-\frac{\pi\ell}{\alpha}]$ where ℓ is a posive integer [27]. Requiring that the growth rate is real and $\hat{\gamma} \ll 1$ we must have $0 < \alpha \ll 1$ with α real so that the stability condition reads:

$$P_0 = \frac{2\lambda\beta_T(r_s/a)^\lambda}{[S^2 - \lambda^2\Omega_0^2]} < \tfrac{1}{4}. \tag{61}$$

Note that the expression above can be rewritten in a more familiar form:

$$\frac{2r_s p_0'(r_s)/B_z^2}{[S^2 - \lambda^2\Omega_0^2]} + \frac{1}{4} > 0, \tag{62}$$

which recovers the usual Suydam instability criterion for $\Omega_0 \to 0$. In the problem of highly localised modes we can relax the constraint (38) allowing $\lambda\Omega_0 > S$. In this case we see from (62) that a complete stabilisation of the mode is expected for monotonically decreasing pressure profiles and sufficiently large flow shear.

2. $m > 1$, $m \sim \mathcal{O}(1)$ internal kink

For the case $m > 1$ with m finite and of order of unity, we focus our attention on the case for which $s \to 0$, i.e. vanishing plasma pressure. In this case we impose the constraint (38), i.e. $S > \lambda\Omega_0$, so that $H > 0$ (see Eq. (41)). We still approximate $D \sim 0$ which corresponds to the limit of vanishing magnetic perturbation at infinity. If this is the case, in (59) we must perform the substitution $\Delta_{c,p} \to \hat{\Delta}_{c,p}$. Since $\Delta_e \to \infty$ and $\hat{\Delta}_{c,p}$ are finite, the modified dispersion relation (59) reduces to:

$$\hat{\gamma} = -\frac{\pi}{\hat{\Delta}_c + \hat{\Delta}_p}. \tag{63}$$

Thus the instability condition is $(\hat{\Delta}_c + \hat{\Delta}_p) < 0$. Since $(\hat{\Delta}_c + \hat{\Delta}_p) \sim \cot \pi\xi$ (see the next section for further details) the stability boundary is identified by the relation:

$$m - \bar{m} = -\lambda\ell, \tag{64}$$

where ℓ is a positive integer. Considering the case where λ is generic and $\delta = 0$, the solution of the equation above reads $H = -1 + \ell^2 + \frac{2}{\lambda}(m\ell - 1)$. Since $dH/d\ell > 0$ the minimum value of H defining the stability boundary is:

$$(H)_{min} = \frac{2}{\lambda}(m-1), \tag{65}$$

and instability occurs when $H > (H)_{min}$ with growth rate given by (63).

3. $m = 1$ internal kink

For the $m = 1$ case, we consider again $s \to 0$, i.e. we want to investigate shear flow driven instabilities and as before we assume $S > \lambda\Omega_0$ in order to have H positive (cf. (41)). Let us assume that $0 < H \ll 1$ [27], thus we have:

$$\hat{\Delta}_c \approx -\frac{2}{\lambda}\frac{(\lambda+1)(\lambda+2)}{H},$$
$$\hat{\Delta}_p \approx (2+\lambda)[\ln\lambda + \gamma_E + \Psi(\tfrac{2}{\lambda}) + \tfrac{\lambda}{2}] - 2$$
$$-\frac{\lambda^2 F(1, 1+\frac{2}{\lambda}; 3+\frac{2}{\lambda}, (r_s/a)^\lambda)}{2(\lambda+1)(1-(r_s/a)^\lambda)}\left(\frac{r_s}{a}\right)^{2+2\lambda},$$

where γ_E is the Euler gamma [36], so that $0 < \hat{\Delta}_p \ll -\hat{\Delta}_c$. The growth rate of the $m = 1$ mode is thus given by:

$$\hat{\gamma} = -\pi/\hat{\Delta}_c = \frac{\pi\lambda H/2}{(\lambda+1)(\lambda+2)}, \tag{66}$$

which shows that $H = 0$ identifies the stability boundary.

Note that expressions similar to (65) and (66) have been derived in Ref. [27], where instead of the flow shear the drive of the instability are the effects linear wrt the magnetic shear.

V. RESISTIVE TEARING MODES

In the analysis of resistive instabilities, a small amount of resistivity (denoted by η) is allowed. Hence (5) is modified in the following manner:

$$(\partial_t - \eta\nabla^2)\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{67}$$

Finite resistivity corrections play a role only in the region very close to the resonant surface. Thus as for the problem for ideal modes, we distinguish two regions. An external region where we drop inertial and resistive corrections and an internal region where resistivity and inertia are retained. The treatment of the external region reduces to the one already analysed in the previous sections.

We note that in the inertial-resistive layer we introduce an effective growth rate $\gamma_* = \gamma - inr_s\Omega_s' x$ where as usual $x = (r - r_s)/r_s$ and we recall that $r_s\Omega_s' = -\lambda\Omega_0$. In line with the previous section, we assume that the imaginary part of the growth rate is negligible. Under the assumption that the radial derivatives are dominant, we write the perturbation of the radial magnetic field as:

$$(\gamma - inr_s\Omega_s' x - \frac{\eta}{r_s^2}\frac{d^2}{dx^2})\tilde{B}^r = \boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\tilde{v}^r. \tag{68}$$

Assuming that $\tilde{v}^\vartheta \gg \tilde{v}^r$ (which can be deduced from the ϑ contravariant projection of (67)), the vorticity equation can be written as:

$$mr_s\mu_s' x\left[\frac{d^2\psi}{dx^2} - m^2\psi\right] - m\frac{d(RJ_z/B_0)}{dx}\psi = r_s^2\frac{\rho R^2}{iB_0}\frac{d(\gamma_*\tilde{v}^\vartheta)}{dx}. \tag{69}$$

For sake of simplicity we assume that $r_s\Omega_s'$ and x in the layer are small so that we can approximate $\gamma_* \sim \gamma$, viz. the contribution due to the flow shear is negligible. This is a reasonable assumption since we assume that the perturbations considered have $m \sim \mathscr{O}(1)$. Hence in absence of extreme flow shear at r_s we expect the eigenfunction to be more sensitive on the global shape of the flow profile. If this is the case we can suppose that in the layer the perturbed velocity is essentially perpendicular to the z direction. Thus equations (68) and (69) reduce to the standard equations describing the tearing modes with no flow as the ones analysed in Ref. [37]. Therefore we expect a growth rate with a dependence of the type $\gamma \sim (\Delta')^{4/5}$. We stress the point that previous works focused on the effects of the flow shear in the layer, while the modification of the external Δ' was neglected [21] (a recent treatment of sheared rotation effects in the resistive layer can be found in Ref. [38]). In our case instead, we drop flow shear effects in the layer and we analyse solely the external modification of the eigenfunction, i.e. we provide an expression for Δ'.

As in the case of the ideal highly localised modes we are allowed to relax the constraint (38) and we let $S < \lambda\Omega_0$ (flow shear stronger than the magnetic shear). For the description of the ideal region we use a simplified version of Eq. (33) in which we set $P_0 = \delta = \omega = 0$:

$$\frac{1}{r}\frac{d}{dr}\left[r^3(\omega_A^2 k_{||}^2 - (n\Omega)^2)\frac{dX}{dr}\right] - \omega_A^2(m^2-1)(m\mu - n)^2 X + (n\Omega)^2(m^2-1)X - 2r^2n^2\Omega\Omega' X = 0. \tag{70}$$

Note that the equation above can be obtained by assuming that the perturbations of the axial magnetic field are vanishing in zero β approximation (this is shown in Appendix A). The parallel wave vector and axial flow profiles are given by (36) and (37) respectively. The solution of equation (70) regular on the magnetic axis is given by (42) for $r < r_s$ and by (46) for $r > r_s$ [27, 35] where $s = 0$ and $\bar{m}$ is given by (44) (the quantities $\bar{m}$, ξ, ζ and H have been defined in section IV A where here

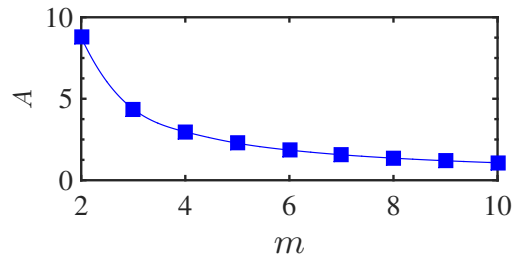

Figure 3. Behaviour wrt the mode number m of the quantity A defined in Eq. (73) for a parabolic axial flow profile and parallel wave vector ($\lambda = 2$). It can be shown that $A >$ also for $\lambda \neq 2$. Since $A > 0$, we conclude from Eq. (72) that the effect of a weak sheared plasma flow is destabilising.

we consider the case of λ generic and $\delta = 0$). For sake of simplicity we set $D \approx 0$ [27, 35].

From (A7) we write $\tilde{\psi}_m \sim z^{1/\lambda}(1-z)X$, thus making use of (50) the tearing stability parameter $\Delta' = \tilde{\psi}_m'/\tilde{\psi}_m|_{r_s-\epsilon}^{r_s+\epsilon}$ [39] is given by [27, 35]:

$$\Delta' = \hat{\Delta}_c + \hat{\Delta}_p = -\frac{m^2 - \bar{m}^2}{r_s\lambda}\pi\cot\pi\xi, \tag{71}$$

where $\hat{\Delta}_c$ and $\hat{\Delta}_p$ are given by (51) and (52) respectively. We write $H = 2\frac{y^2}{\lambda^2}/(1-y^2)$ where $y = \lambda\Omega_0/S$ measures the ratio between the flow and the magnetic shear at the resonant point r_s. We consider the limiting cases of weak ($0 \leqslant y \ll 1$) and strong ($y \gg 1$) flow shear. We focus on the particular case of a parabolically distributed profiles (current density, equilibrium flow and pressure) i.e. $\lambda = 2$ (the conclusions obtained remain valid also when $\lambda \neq 2$).

When the flow shear is small, Δ' can be approximated by:

$$r_s\Delta' \approx r_s\Delta_0' + \pi A y^2, \tag{72}$$

$$A = \left[\tfrac{\pi\bar{m}_0}{4}(1 - (m/\bar{m}_0)^2)\csc^2\pi\xi_0 + \cot\pi\xi_0\right], \tag{73}$$

$$r_s\Delta_0' = 4\pi\cot\pi\xi_0 \tag{74}$$

where $\xi_0 = (m - \bar{m}_0)/2$ and $\bar{m}_0 = \sqrt{m^2+8}$. The quantity A as defined in (73) is numerically found to be positive as shown in figure 3, indicating the destabilising effect of the weak flow shear.

Conversely when $y \gg 1$ the tearing stability index is given by:

$$r_s\Delta' = 3\pi\cot\pi\xi_* \tag{75}$$

where $\bar{m}_* = \sqrt{m^2+6}$ and $\xi_* = (m - \bar{m}_*)/2$. In this particular case we find that $\Delta' < \Delta_0'$ which indicates that the mode is stabilised. This is shown in Fig. 4.

The full behaviour of Eq. (71) against the parmeter y, viz. the ratio between flow and magnetic shear, is

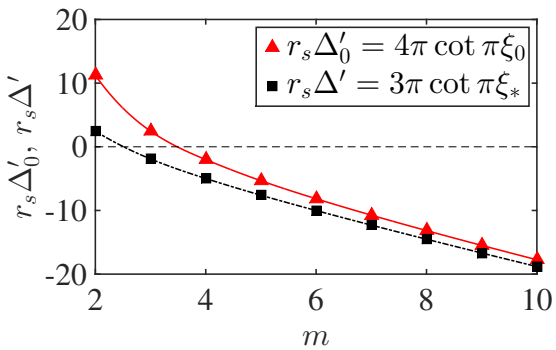

Figure 4. Behaviour wrt the mode number m of the tearing stability given by Eqs. (74) (red triangles) and (75) (black squares) for the parabolic case ($\lambda = 2$). For large flow shear, the tearing stability index is diminished wrt the case where flow shear is absent. Thus for this case we expect mode stabilisation for each m.

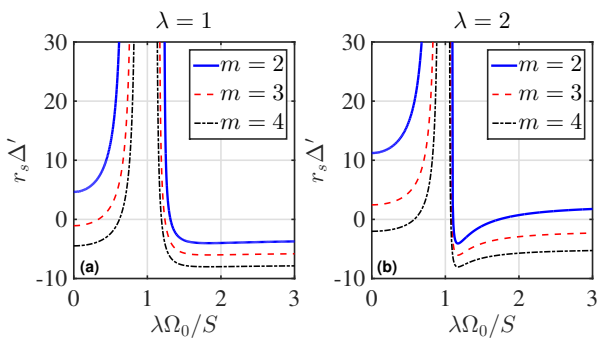

Figure 5. Behaviour of (71) for $\lambda = 1$ (a) and $\lambda = 2$ (b) wrt $\lambda\Omega_0/S$ for different poloidal mode numbers with $n = 1$. It is evident that small flow shear has a destabilising effect, while a large flow shear tends to diminish the total Δ' and opens intervals of stability. We omitted to plot Δ' in the region lying between the first and the last zero of $\sin \pi\xi$ (i.e. when the parameter $\lambda\Omega_0/S$ is near unity) since $r_s\Delta'$ is either highly oscillating or not defined.

shown in Fig. 5 where the $\lambda = 1$ and $\lambda = 2$ cases are considered.

In TCV, DIII-D and NSTX dicharges [6, 40, 41], the ratio between rotation shear and magnetic shear is usually smaller than unity assuming values around $10^{-2} \div 10^{-3}$, thus we might expect a weak destabilisation. However toroidal effects may play an important role in the modification of the stability boundaries. Hence a more complicated behaviour is expected in toroidal geometry [25].

If finite β effects are allowed, the expression the tearing stability index is [23, 42]:

$$r_s\Delta' = \Delta_c + \Delta_p, \tag{76}$$

where Δ_c and Δ_p are defined by (48) and (49) respectively. In the parabolic case ($\lambda = 2$), for small s, i.e. small β corrections, we have (cf. Ref. [23]):

$$r_s\Delta' = r_s\Delta'[1 + (\tfrac{1}{\zeta} - 2 + 4\gamma_E + 2\ln 2 + \Psi(\xi) + \Psi(-\xi) + 2\Psi(\zeta))s], \tag{77}$$

where $r_s\Delta'$ is given by (71) (obviously the quantities ξ and ζ must be evaluated with $\lambda = 2$). We note that an analogous expression can be found for $\lambda \neq 2$.

We can now estimate the impact of the flow shear on the $m = 1$ mode. Since we want to connect with the notation in Ref. [17] we drop the normalisation $\omega_A = 1$. Assuming that finite β effects are negligible, and that the flow shear corrections affect essentially the external eigenfunction, from Ref. [17] we write the dispersion relation as:

$$-\frac{\hat{\Delta}_c}{\pi}\epsilon_\eta^{1/3} = \frac{h^{5/4}}{8}\frac{\Gamma[(h^{3/2}-1)/4]}{\Gamma[(h^{3/2}+5)/4]},$$

where $\hat{\Delta}_c$ is given by (51), $h = \gamma/(\omega_A\epsilon_\eta^{1/3})$ and $\epsilon_\eta = 1/\omega_A\tau_R$ with $\tau_R = r_s^2/\eta$. Assuming $0 < H \ll 1$ (small flow shear) a series expansion in H gives [17]:

$$h = 1 + \frac{2\sqrt{\pi}}{3\epsilon_\eta^{1/3}}\frac{\lambda H}{(\lambda+1)(\lambda+2)}, \tag{78}$$

which shows the destabilising effect of the sheared axial flow.

Because of the strong localisation, we might expect modes with $m \gg 1$ to be more sensitive to the flow shear corrections in the resistive layer. This problem however is not treated here and it will be studied in a future work.

VI. CONCLUSIONS

In this work we investigated the impact of an equilibrium sheared axial flow on both ideal and resistive MHD instabilities in cylindrical geometry. A Newcomb's like equation has been derived which includes the equilibrium flow shear (i.e. the Velikhov term) and finite β effects.

Following the standard procedure for the description of MHD perturbations, the analysis is split into two regions. Choosing a particular Fourier mode, in the outer region far from the resonant point r_s identified by $k_{||} = 0$, inertia and resistivity are neglected while equilibrium shear flow effects are retained. The inner layer region, lying in a neighbourhood of r_s, is augmented by inertia and resistivity corrections (for ideal modes we let resistivity go to zero).

In order to simplify the analysis, the computations have been carried out in a reference frame such that the axial flow vanishes at the resonant surface of the mode considered. With a careful choice of the profiles

of the equilibrium flow, rotational transform and pressure profiles the equation describing the generalised radial fluid displacement far from the resonant layer (cf. Eq. (33)) has been exactly solved (cf. Eqs. (42) and (46)).

In the ideal frame, the solution in the inner inertial layer has been calculated (cf. Eqs. (54) and (55)) and close to marginal stability it has been matched with the outer region solution. Thus a dispersion relation has been derived and analysed for three different limits of the poloidal mode number m. Summarising, our analysis allowed us to find a stability criterion for $m \gg 1$ Suydam-like modes in agreement with previous results [8, 11]. Beyond this, instability thresholds in the flow shear parameter H (cf. (41)) have been found for $m > 1$ (internal kinks with $m \sim \mathscr{O}(1)$) and $m = 1$ modes. It is found that a weak shear flow has always a destabilising effect, while a complete stabilisation for highly localised ($m \gg 1$ Suydam-like) modes occurs for strong shear flows larger than the magnetic shear at the mode resonant point for monotonically decreasing pressure profiles. Strong shear flow effects on internal kink modes ($m \geq 1$ with $m \sim \mathscr{O}(1)$) have not been assessed and they will be investigated in future work.

In the resistive frame the behaviour of tearing-like instabilities has been investigated. We neglected flow shear effects in the inertial-resistive layer, i.e. we assessed the impact of flow shear only on the external eigenfunction. With the same class of profiles for the plasma flow and rotational transform, a novel and exact expression for the tearing stability index Δ' has been derived (cf. Eq. (71)). It is found as in the case of ideal Suydam-like modes, that a weak flow shear has a destabilising effect (increasing Δ'), while a strong flow shear tends to stabilise the mode (diminishing Δ'). In some particular cases Δ' can change sign from positive to negative indicating a complete stabilisation. Finite β corrections have also been included in the expression of Δ' (cf. Eq. (77)). Finally a brief discussion on the $m = 1$ mode has been presented, where it is found that a weak flow shear has a destabilising effect.

Future work will be focused on extending the analysis to a fully toroidal geometry, in particular studying the effects of plasma flows on the nonlinear dynamics (evolution and saturation) of resistive modes with a generalised Rutherford equation possibly with the inclusion of neoclassical effects [43]. It is envisaged that this work could help providing physical guidelines for control techniques of such perturbations.

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Appendix A: Reduced Newcomb equation

Here we derive a simplified Newcomb equation, which is suitable for tearing mode analysis. The geometry considered has been already described in Sec. II. As before the equilibrium velocity is $\boldsymbol{v}_0 = \Omega(r)\boldsymbol{e}_\varphi$ and we assume that the mass density and the toroidal magnetic field are constant in space and time, i.e. $\varrho = \varrho_0$ and $B^\varphi = B_0^\varphi$. Note that $B^\varphi = B_0/R$ (see Section II). The perturbed magnetic field is thus written as $\boldsymbol{B} = -\boldsymbol{\nabla}\tilde{\psi} \times \boldsymbol{\nabla}\varphi$. The physical model is given by Eqs. (4)-(7). Neglecting time derivatives, we multiply (4) by $1/B^\varphi$ and then we take the $\boldsymbol{\nabla}\varphi$ projection of its curl, leading to (finite β effects are neglected):

$$\frac{\varrho}{B^\varphi}\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla} \times (\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v}) = \boldsymbol{B} \cdot \boldsymbol{\nabla}\left(\frac{J^\varphi}{B^\varphi}\right). \tag{A1}$$

Similarly to equation (27), linearisation of Eq. (A1), gives:

$$-n\frac{\varrho R^2}{B^\varphi}\left(\Omega\tilde{w}^\varphi + \frac{r}{R}\Omega'\tilde{v}^\vartheta\right) = (m\mu - n)\left[\frac{1}{r}(r\tilde{\psi}')' - \frac{m^2}{r^2}\tilde{\psi}\right] - \frac{m}{r}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{q}\right)\right]\tilde{\psi}. \tag{A2}$$

In order to obtain the perturbed velocities, we make use of the Faraday-Ohm's law $\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) \approx 0$ which can be written by components in the following form:

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} v^i - \boldsymbol{v} \cdot \boldsymbol{\nabla} B^i = 0. \tag{A3}$$

By means of $(\sqrt{g}\tilde{B}^r)_m = -im\tilde{\psi}_m$ and $(\sqrt{g}\tilde{B}^\vartheta)_m = \tilde{\psi}'_m$, the linearised (m, n) Fourier component of (A3) reads:

$$\tilde{v}^r = \frac{imn\Omega}{F_0'(m\mu - n)}\tilde{\psi}_m, \tag{A4}$$

$$\tilde{v}^\vartheta = -\frac{1}{imr}\left[(r\tilde{v}^r)' - \frac{r\Omega'}{\Omega}\tilde{v}^r\right], \tag{A5}$$

$$\tilde{v}^\varphi = \frac{m\Omega'}{F_0'(m\mu - n)}\tilde{\psi}_m, \tag{A6}$$

where the velocity fulfils the constraint $\boldsymbol{\nabla} \cdot \tilde{v} = 0$. We note that in absence of flow shear we do not have a toroidal perturbed velocity and $\tilde{v}^\vartheta = -1/(im)[(\tilde{v}^r)' + \tilde{v}^r/r]$. In terms of the quantity X, (A4) becomes:

$$\tilde{\psi}_m = -\frac{rB_0k_{||}}{m}X. \tag{A7}$$

After some lenghty but straightforward algebra, one can write the lhs of (A2) as (cf. Ref. [30] pages 262-263):

$$-n\frac{\varrho R^2}{B^\varphi}\left(\Omega\tilde{w}^\varphi + \frac{r}{R}\Omega'\tilde{v}^\vartheta\right) = -\frac{n^2\varrho R^2}{mB_0} \times \times\left\{{}^1\!/_{r^2}\left(r^3\Omega^2X'\right)' + {}^1\!/_r(1-m^2)\Omega^2X + 2\Omega\Omega'X\right\}. \tag{A8}$$

Combining Eqs. (A2) and (A8) we arrive at the Newcomb equation in presence of an equilibrium axial flow (cf. Ref. [30] pages 262-263):

$$\frac{1}{r^2}\frac{d}{dr}(r^3[k_{||}^2-(n\frac{\Omega}{\omega_A})^2]\frac{dX}{dr})+ [\frac{1}{r}(1-m^2)(k_{||}^2-(n\frac{\Omega}{\omega_A})^2)-2\frac{n^2\Omega\Omega'}{\omega_A^2}]X=0,$$

which is equation (70).

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